ABSTRACT BILINEAR INTEGRATION AND APPLICATIONS

By

JASON T. KOZINSKI

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Jason T. Kozinski

I dedicate this work to my wife Gayle for her love, patience, and support over the past three years, and to my cat Hyzenthlay for her joyful spirit during the difficult times.

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By

Jason T. Kozinski

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Chair: James K. Brooks
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This study develops a bilinear integral and Lebesgue space for operator-valued functions H with respect to an infinite-dimensional measure m, all taking values in abstract locally convex spaces. The integral is then applied to the development of a stochastic integral for general processes H with respect to a square-integrable martingale X.

First, a semivariation is defined in the Banach space setting that more readily lends itself to the construction of a bilinear integral for operator-valued functions. A bilinear integral and Lebesgue space are defined in this context using determining sequences, and is then applied to the setting of the stochastic integral.

Next, a bilinear integral and Lebesgue space, including convergence theorems, are constructed in the more general setting of locally convex space. Under certain assumptions, the bounded measurable functions are integrable. An integral for bounded functions in particular is given through an alternate approach that agrees with the previous integral under the prior assumptions. The spaces L^1_G and L^2_G are developed for a locally convex space G.

Finally, a stochastic integral and Lebesgue space are developed for processes H with respect to a square-integrable martingale X in the setting of nuclear locally convex space under certain assumptions.

CHAPTER 1
INTRODUCTION

During the last half-century, the theory of stochastic integration has become a topic of extensive international study as a result of its benefit to the field of probability and significant physical applications. The development of the theory is in fact due to its utility to the study of Brownian motion and stochastic differential equations. An abundance of research and numerous texts have followed the early work of Ito, including a thorough exposition of real stochastic integration by Mayer, and a generalization to Hilbert-valued processes by Kunita, Metivier, and others. Various approaches have been put forth to extend the notion to Banach-valued processes. One such idea, of Pellaumail, is to represent the stochastic integral as a genuine integral with respect to an L^1-valued "stochastic measure" on a σ-algebra of subsets of $\mathbb{R}^+ \times \Omega$ where $(\Omega, \Sigma, \mathcal{P})$ is the underlying probability space. The lack of a satisfactory integration theory for infinite-dimensional vector measures made this goal impossible. Other constructions, produced most notably by the French school, either have fallen short of needed generality within spaces, or have lacked the crucial convergence theorems one would expect in order to fully develop a theory of stochastic integration.

Within the last decade, a general theory of stochastic integration in Banach space has been developed by Brooks and Dinculeanu [B–D3], by first fashioning a bilinear vector integration theory with respect to an operator-valued measure of finite semivariation [B–D2]. Given a process $X : \mathbb{R}^+ \times \Omega \to E \subset L(F, G)$, where the spaces are Banach and where X is L^2_E-valued for each fixed $t \geq 0$, the integral is defined with respect to an L^2_E-valued measure I_X on the predictable σ-algebra by regarding $L^2_E \subset L(F, L^2_G)$. This construction yields a complete seminormed Lebesgue-type space of I_X-integrable processes.

A treatment of the stochastic integral for certain nuclear-valued square-integrable martingales has been given by Ustunel [U] and more generally by Brooks and Neal [B–N]. In this setting, the range of certain bounded integrands proves to be absorbed into a

Hilbert space that is continuously embedded in the nuclear space E, and the stochastic integral is defined for certain E'-valued processes X. The general case, in which one has $X : \mathbb{R}^+ \times \Omega \to L(F,G)$ and the spaces are nuclear, has very recently been developed by Brooks ([Br2],[Br3]), employing a bilinear integration theory for certain processes X and the familiar stochastic measure I_X on the predictable σ-algebra by regarding $L^2_E \subset L(F, L^2_G)$.

Recent physical applications have given rise to a need for the consideration of a stochastic integral for operator-valued processes. For example, it is desirable to consider a stochastic integral of processes $H : \mathbb{R}^+ \times \Omega \to E'$ and $X : \mathbb{R}^+ \times \Omega \to E$ where E is a nuclear space (or more generally $H : \mathbb{R}^+ \times \Omega \to F \subset L(E,G)$ where the spaces are nuclear). The treatment of this problem by utilizing the theory of Brooks places objectionable complexity upon the resulting infinite-dimensional vector measure I_X. A more suitable approach would be to shift some of this complexity onto H, to which, given a suitable definition of a Lebesgue-type space on the integrable functions, powerful convergence and compactness theorems may be applied. We wish to do so by regarding $E' \subset L(L^2_E, L^2_{\mathbb{R}})$ so that the range of H operates upon the L^2_E-valued measure I_X in a natural way. This idea, however, necessitates the development of a bilinear integration theory for operator-valued functions with respect to an infinite-dimensional measure, the construction of a Lebesgue-type space, and theorems for convergence and compactness in such a space from which theorems for the convergence of integrals would naturally arise.

The above considerations form the motivation for the work presented herein. The first goal of this study is to construct a theory of bilinear integration and Lebesgue space for operator-valued functions with respect to an infinite-dimensional measure. Secondly, we wish to develop a bilinear integration and Lebesgue space for the case of processes $H : \mathbb{R}^+ \times \Omega \to E'$ and $X : \mathbb{R}^+ \times \Omega \to E$ where E is a nuclear space satisfying the special conditions of Ustunel. Of greatest importance, we would like the integration theory to include the possibility of unbounded integrands, which were not considered in the work of Ustunel nor in that of Brooks and Neal. Finally, we would like to extend these ideas to more general locally convex spaces in a meaningful way. Each of these goals is addressed and met within this text, and the process of exploring the particular problems has led to significant generalizations of the bilinear integral to a larger class of locally convex spaces, with the potential for further development. We now give a brief description and point to the significance of the content of each chapter.

Chapter 2 outlines the definition and properties of a semivariation more suited to the purpose of creating an integral for operator-valued functions, and which is used throughout the remaining chapters. It is defined for an additive F-valued measure $\boldsymbol{m}$ relative to a continuous linear map $\phi : E \to L(F, G)$, where E, F, G are Banach spaces. The exploration of this semivariation in the setting of operator-valued functions reveals new insights into the potential definition of negligible sets and measurability conditions for integrable functions, even for the case of operator-valued measures, when generalizing from an isometry to a continuous embedding into the space of operators. The semivariation developed in the setting of operator-valued functions places greater emphasis on the map ϕ, and a relationship between the two notions of semivariation is given that more properly illustrates the symmetry and bilinear nature of the action of the spaces involved. In particular, any ϕ-semivariation may be realized as a ψ-semivariation and conversely by way of a particular continuous bilinear map, although there are important distinctions between the two. Most notably, the ϕ-semivariation of a measurable set A may be 0 without the same being true of $\boldsymbol{m}(A)$, even when ϕ is an isometry.

In chapter 3, a bilinear integration and Lebesgue space are created from the point of view of operator-valued functions, that is, for E-valued functions with respect to an F-valued measure under the influence of a continuous linear map $\phi : E \to L(F, G)$, where E, F, G are Banach spaces. The construction presented here differs from the one provided of operator-valued measures in that the integrable functions and the integral are defined in terms of the more classical notion of determining sequences and convergence in G, rather than being defined as an element of the bidual G''. Additionally, the definitions of negligible sets and measurability reflect a greater emphasis on ϕ, avoiding some potential pitfalls that are discussed in chapter 2. The application to the stochastic integral is also examined by considering the continuous linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$ and the set function $I_X : \mathscr{P} \to L^2_F$ defined in the usual manner for a square-integrable martingale

$X : \mathbb{R}^+ \times \Omega \to F$ on the predictable σ-algebra $\mathscr{P}$, with results similar to those of Brooks and Dinculeanu.

Chapter 4 contains the main body of the text. The underlying concepts of the chapter come from an observation of the nuclear E, E' case of the stochastic integral. Given a projective limit of square-integrable martingales in E, one may find a limit $X : \mathbb{R}^+ \times \Omega \to E$, as shown by Brooks and Neal [B–N], which may additionally be assumed to take its values in the space $E_o = E[V^\circ]$ for some closed convex balanced neighborhood V of E', and where V° is the polar of V. Given another process $H : \mathbb{R}^+ \times \Omega \to E'$, consider the continuous linear map $\phi : E' \to L(L^2_{E_o}, L^2_{\mathbb{R}})$. In defining a stochastic integral $H \cdot X$, it seems that one only need consider $(E_o)'$, which is equal to $E'(V)$ if E satisfies the special conditions of Ustunel. Indeed, one may use the ϕ_o-semivariation relative to $\phi_o : (E_o)' \to L(L^2_{E_o}, L^2_{\mathbb{R}})$ in order to construct an integral and Lebesgue space of integrable processes. This idea can be seen to extend to more general nuclear spaces E, F, G under a continuous linear map $\phi : E \to L(F, G)$, which induces (as one can verify) a continuous linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$. This, of course, necessitates a definition of what is meant by L^2_G when G is locally convex, and the construction of a bilinear integration theory for functions and measures taking values in locally convex space, as each of E, L^2_F, L^2_G will be locally convex. That is, we need a bilinear integration with respect to a map $\phi : E \to L(F, G)$, where E, F, G are locally convex spaces. Section 4.2 addresses this question in the event that F is Banach, which is suitable for our purposes. For, if F satisfies the special conditions of Ustunel and the measure $\boldsymbol{m}$ is σ-additive, then section 4.3 shows that we may reduce to the Banach setting by observing that the range of $\boldsymbol{m}$ is bounded. On the other hand, if one is interested in the stochastic setting and the measure I_X comes from the limit X of a projective system of square-integrable martingales, then as mentioned above, one may assume that L^2_F is a Banach space. In order to develop the integration and Lebesgue space for locally convex spaces E, G and Banach space F, one uses the idea of the E, E' case to motivate the definition of the reducing seminorms of section 4.1. These seminorms "reduce" the map $\phi : E \to L(F, G)$ to isometries $\phi_r : E(U_p) \to L(F, G(U_r))$ between Banach spaces for suitable continuous seminorms p, r on E, G respectively. The locally convex Lebesgue space of integrable functions and its seminorms are then built upon the resulting ϕ_r-semivariations. The bounded measurable functions are shown to be integrable in the event that G has a countable basis and $G(U_r) \not\supset c_o$ for each continuous seminorm r in a basis of continuous seminorms for G. Note that these conditions are satisfied when $G = L^2_{\mathbb{R}}$, and thus the construction encompasses the previous E, E' nuclear setting, and extends to also include integration of potentially unbounded processes. In section 4.4, a more general alternative integral is obtained for bounded functions when certain conditions are placed on E, F, G by examining a continuous linear map $\phi : E[B_o] \to L(F, G[B])$, where B_o, B are closed convex balanced bounded sets of E, G respectively, and for which B_o absorbs the range of H. This alternative integral is shown to agree with the definition of the integral under the conditions in the previous setting. It should be noted that these alternative integrals generally require the development of Banach space integration with respect to nonisometric embeddings as outlined in chapter 3. The most interesting technique that is employed in chapter 4 is to combine theorems of Pettis with the special conditions of Ustunel in order to obtain relationships between E and the spaces $E[B]$, which are contained in E, but have a stronger topology (see theorem 4.3.2 or lemma 4.4.1 for examples).

In order to prepare for the stochastic integral of chapter 6, the content of chapter 5 concentrates on the definition of the space $L^2_G(\boldsymbol{m})$, where G is a locally convex space and $\boldsymbol{m}$ is a scalar measure of finite total variation $|\,\boldsymbol{m}\,|$. It is defined in terms of $L^1_G(\boldsymbol{m})$ in a fashion similar to that of the classical Banach setting, but with respect to each seminorm r in a basis of continuous seminorms for G. It is verified, as one would hope, that the space L^1_G is seen to be a special case of the integration outlined in chapter 4, by way of the continuous linear map $\phi : G \to L(\mathbb{F}, G)$ (where $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$ depending on the context), which reduces, upon application of a continuous seminorm r, to $\phi_r : G(U_r) \to L(\mathbb{F}, G(U_r))$. Although this is done in all generality, we will be particularly interested in a probability measure $\mathcal{P}$. Additionally, we wish to ensure that L^2_G is complete, so that the stochastic integral of a process H, constructed in chapter 6, will be function-valued, as its definition will be as a limit of a Cauchy sequence in L^2_G. This is shown to be the case when G has a countable basis of seminorms. At its conclusion, the chapter provides some technical relationships among the spaces L^2_G, $L^2_{G(U)}$, and $L^2_{G[B]}$ that are crucial in order to define the stochastic integral.

Finally, chapter 6 contains the development of the stochastic integral for integrable processes $H : \mathbb{R}^+ \times \Omega \to E$ relative to the continuous linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$, defined in the obvious manner from a given continuous linear $\phi : E \to L(F, G)$, for a Banach space F and locally convex spaces E, G satisfying the special conditions of Ustunel. The underlying measure I_X comes from a square-integrable martingale as usual. It is nontrivial to prove that the map $\overline{\phi}$ is indeed continuous and linear, due to the fact that the spaces are locally convex and since L^2_G is newly defined. The relationships between L^2_G-type spaces, proven laboriously at the end of chapter 6, will guarantee that the ϕ_r-semivariations will be finite in this setting. When G (and hence L^2_G) has a countable basis of seminorms, one may apply the integration theory of chapter 4 to this setting in order to create a Lebesgue space of integrable processes for which the typical convergence theorems hold. The definition of a square-integrable martingale is extended in a natural way for processes with values in locally convex space, and with this definition, given any integrable process H, the process defined for $t \geq 0$ by

$$(H \cdot X)_t \ := \ \int H 1_{[0,t]} \, \mathrm{d}X \ := \ \int H 1_{[0,t]} \, \mathrm{d}I_X$$

is a square-integrable martingale.

CHAPTER 2
SEMIVARIATION

In this chapter, we first recall the definition of the usual semivariation, and then define a new semivariation, which is more appropriate for the purpose of developing an integration theory for operator-valued functions. This semivariation will extend to include the possibility that the mapping of Banach spaces E into $L(F, G)$ is merely continuous and linear, rather than an isometry, where the measure under consideration is F-valued. Some errors and omissions related to the classical treatment of nonisometric embeddings in the setting of operator-valued measures are addressed, and examples are given. One may refer to the text of Dinculeanu [D1] for a more thorough discussion of the usual semivariation.

2.1 The Usual Semivariation

We first recall the definition of the usual semivariation, and how it is used to create a Lebesgue space of integrable functions. Unless otherwise stated, all spaces to be considered will be taken over either the real or complex number field. Let E, F, G be Banach spaces for which there is an *isometry* $\psi : F \to L(E, G)$. We let $\boldsymbol{m} : \Sigma \to F$ be an additive measure on the σ-algebra Σ of subsets of some set T. The *usual semivariation* of $\boldsymbol{m}$ with respect to E, G is defined to be

$$\widetilde{\boldsymbol{m}}_{E,G}(A) \;=\; \sup \| \sum_{i=1}^{n} \boldsymbol{m}(A_i) e_i \|_G \quad \text{for} \quad A \subset T,$$

where the supremum is taken over $e_i \in E_1$, the unit ball of E, and all finite collections $\{A_i\} \subset \Sigma$ of pairwise disjoint subsets of A, which we will subsequently denote $A_i \amalg A$. The usual semivariation is monotone and finitely subadditive on Σ, σ-subadditive when $\boldsymbol{m}$ is σ-additive, and satisfies

$$\| \boldsymbol{m}(A) \|_F \;\le\; \widetilde{\boldsymbol{m}}_{E,G}(A) \;\le\; | \boldsymbol{m} |(A) \quad \text{for} \quad A \in \Sigma \,, \tag{2.1}$$

where $|\boldsymbol{m}|$ is the total variation of $\boldsymbol{m}$. A crucial property of the semivariation is the computation

$$\widetilde{\boldsymbol{m}}_{E,G}(A) \;=\; \sup_{z\in(G')_1} |\boldsymbol{m}_z|(A) \quad \text{for } A \in \Sigma\,, \tag{2.2}$$

where $\boldsymbol{m}_z : \Sigma \to E'$ is defined for $z \in G$ by

$$\boldsymbol{m}_z(A)e \;=\; z\boldsymbol{m}(A)e \quad \text{for } e \in E \text{ and } A \in \Sigma\,.$$

Each $\boldsymbol{m}_z$ is an additive measure on Σ (σ-additive when $\boldsymbol{m}$ is σ-additive). If $\widetilde{\boldsymbol{m}}_{E,G} < \infty$, then (2.2) shows that

$$\boldsymbol{m}_{E,G} \;:=\; \{\, |\boldsymbol{m}_z| : z \in (G')_1 \,\}$$

forms a bounded set of positive measures. By utilizing these measures $\boldsymbol{m}_z$ of finite variation, an integral can naturally be defined for functions $H : T \to E$ satisfying certain conditions for measurability.

We say a set $A \subset T$ is $\boldsymbol{m}$-negligible if $\boldsymbol{m}(B) = 0$ for every set $B \in \Sigma$ for which $B \subset A$. Inequality (2.1) and computation (2.2) show that $A \in \Sigma$ is $\boldsymbol{m}$-negligible if and only if $\widetilde{\boldsymbol{m}}_{E,G}(A) = 0$, and hence if and only if A is $|\boldsymbol{m}_z|$-negligible for each $z \in (G')_1$. This will *not* generally be the case if ψ is not an isometry. By the standard convention, we say that a property occurs $\boldsymbol{m}$-almost everywhere ($\boldsymbol{m}$-*a.e.*) if it is valid outside of an $\boldsymbol{m}$-negligible set. For $\boldsymbol{m}$-measurable functions $H : T \to E$, that is, those that are a pointwise limit $\boldsymbol{m}$-*a.e.* of E-valued measurable simple functions, one may define

$$N(H) \;=\; \sup_{z\in(G')_1} \int \|H\|_E \, \mathrm{d}|\boldsymbol{m}_z| \;\le\; \infty\,.$$

Then N is a seminorm on those H for which $N(H) < \infty$, and the space of measurable H for which $N(H) < \infty$, denoted $\mathfrak{F}(\boldsymbol{m}_{E,G})$, is a seminormed space under N. Integration of functions with respect to a scalar measure of finite variation is well-known—the reader is suggested to consult a standard text ([D–S],[H])—and the conditions placed on functions of $\mathfrak{F}(\boldsymbol{m}_{E,G})$ imply that the integrals $\int H\, \mathrm{d}\boldsymbol{m}_z$ exist for each $z \in G'$. The mapping $z \mapsto \int H\, \mathrm{d}\boldsymbol{m}_z$ is an element of G'' and is the definition of the integral $\int H\, \mathrm{d}\boldsymbol{m}$. Of course, the case of most interest is that in which $\int H\, \mathrm{d}\boldsymbol{m} \in G$, which is shown by Brooks and Dinculeanu [B–D3] to be the case under any of the following assumptions:

(1) G is reflexive.

(2) $G \not\supset c_o$.

(3) H is the limit in $\mathfrak{F}(\boldsymbol{m}_{E,G})$ of simple measurable functions.

(4) H is bounded and $\boldsymbol{m}$-measurable and the set $\boldsymbol{m}_{E,G}$ is relatively weakly compact in $ca(\Sigma)$, the space of σ-additive scalar measures on Σ under the total variation norm.

2.2 ϕ- and ψ- Semivariation

In order to facilitate the general development of a bilinear integral for functions $H : T \to E$ with respect to a measure $\boldsymbol{m} : \Sigma \to F$, where E and F are no longer Banach spaces, one would like to extend the concept of the semivariation in two principal directions. First, it is useful to consider the possibility that E, rather than F, is operator-valued, say $E \subset L(F, G)$ under some isometric embedding. Secondly, it is desirable to consider the existence of a map $\phi : E \to L(F, G)$ that is merely continuous and linear, rather than an isometry.

There are a number of motivations toward the inclusion of these considerations. The foremost is that the notion of isometry is useless outside the realm of normed space, particularly in locally convex space. It would make more sense to consider a map that is continuous, linear, and perhaps injective. Secondly, even in the simplest of cases, the integral defined using the usual semivariation can cause unneeded complication of the space in which the measure takes its values. For example, examine the case of a Banach space E and a complex-valued measure $\boldsymbol{m}$. We have an isometry into, $\psi : \mathbb{C} \to L(E, E)$, given by $\psi(\alpha)e = \alpha e$. But, $\phi : E \to L(\mathbb{C}, E)$ is an isometry onto, given by $\phi(e)\alpha = \alpha e$, and we avoid consideration of the more complicated space $L(E, E)$. The same is true for the case of an E'-valued function and E-valued measure. It would seem more natural to consider $E' = L(E, \mathbb{C})$ than to evoke the bidual $L(E', \mathbb{C})$. As one might see from these simple examples, there is a dual nature to considering operator-valued functions or measures in turn, which shall be explored later in this section. More generally, one could consider a continuous bilinear map $\Phi : E \times F \to G$, which induces each of these cases, and thus, there is no cause for discrimination against the notion of operator-valued functions. In any event, the naturally induced mappings will not necessarily be injective (nor isometries, in the Banach case).

It should be noted that, for the case of operator-valued measures over Banach spaces, a generalization to continuous linear maps has been explored ([B–D3],[D2]). However, some of these results contain either errors or a loss of generality that should be addressed, as they render false several statements concerning measurability and alter inequalities that appear in the proofs of some important theorems. Most of the objections arise from the oversight of two properties. First, the inequality (2.1) fails in nontrivial (and even trivial) cases, as it is possible for the usual semivariation of a set $A \in \Sigma$ to be 0, while $\boldsymbol{m}(A) \neq 0$, as seen in examples 2.2.1 to 2.2.3. Thus, in particular, the statement ([D2]) that $A \in \Sigma$ is $\boldsymbol{m}$-negligible if and only if A is $|\,\boldsymbol{m}_z\,|$-negligible for each $z \in (G')_1$ is false, but is used in a number of important proofs that use measurability conditions, such as that of completeness of the space of integrable functions (see theorem 3.1.12 as an example of how it would be applied). Secondly, one cannot in complete generality assume that the continuous linear map $\psi : F \to L(E, G)$ satisfies $\|\,\psi(f)\,e\,\|_G \leq \|\,e\,\|_E \,\|\,f\,\|_F$ for each $e \in E$ and $f \in F$, which is the case when $\|\,\psi\,\|_{F,L(E,G)} \leq 1$. If one does so and uses the convention of writing $\psi(f)$ as f, then it is prudent not to disregard ψ at crucial times, when one may lose sight of certain dependence upon ψ. For example, the inequality

$$\|\,\boldsymbol{m}(A)e\,\|_G \;\leq\; \|\,\boldsymbol{m}(A)\,\|_F \,\|\,e\,\|_E$$

is in actuality

$$\|\,\boldsymbol{m}(A)e\,\|_G \;\leq\; \|\,\psi\,\|_{L(F,L(E,G))} \,\|\,\boldsymbol{m}(A)\,\|_F \,\|\,e\,\|_E \;,$$

and in general, there is no reason to expect $\|\,\psi\,\|_{L(F,L(E,G))} \leq 1$. For example, see proposition 3.2.4 and compare to the analogous theorem (Theorem 17.5, [D2]).

We first explore the question of extending the usual semivariation to include continuous, linear maps. Let E, F, G be Banach spaces, and let $\boldsymbol{m} : \Sigma \to F$ be an additive measure defined on a σ-algebra Σ of subsets of some set T. Given any continuous, linear map $\psi : F \to L(E, G)$, we define the ψ-semivariation by

$$\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \;=\; \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \|\sum_{i=1}^{n} \psi(\boldsymbol{m}(A_i))e_i\,\|_G \;\text{ for }\; A \subset T.$$

When ψ is understood, we may simply refer to $\widetilde{\boldsymbol{m}}_{\psi,E,G}$ as the semivariation and write $\widetilde{\boldsymbol{m}}_{E,G}$. Note that the usual semivariation agrees with the ψ-semivariation when ψ is an isometry, since, in that case, each $\boldsymbol{m}(A)$ is regarded as the element $\psi(\boldsymbol{m}(A))$ of $L(E,G)$. The ψ-semivariation, on the other hand, places emphasis on the fact that it is the image of $\boldsymbol{m}(A)$ that is operator-valued, and this difference will alter several results.

As briefly noted above, the principal nuance in dealing with a nonisometry ψ is that the inequality

$$\|\boldsymbol{m}(A)\|_F \;\leq\; \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \;\text{ for }\; A \in \Sigma \tag{2.3}$$

may no longer hold. In the case of an isometry, this inequality follows from the fact that

$$\|\boldsymbol{m}(A)\|_F \;=\; \|\psi(\boldsymbol{m}(A))\|_{L(E,G)} \;=\; \sup_{e\in E_1} \|\psi(\boldsymbol{m}(A))e\|_G \;,$$

and immediately yields that $\boldsymbol{m}(A) = 0$ whenever $A \in \Sigma$ and $\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) = 0$. For a continuous linear ψ, we certainly have that

$$\|\psi(\boldsymbol{m}(A))\|_{L(E,G)} \;\leq\; \|\psi\|_{L(F,L(E,G))}\,\|\boldsymbol{m}(A)\|_F \;,$$

but not the reverse. The following examples demonstrate that we cannot expect to have the inequality $\|\boldsymbol{m}(A)\|_F \leq \widetilde{\boldsymbol{m}}_{\psi,E,G}(A)$ in general.

Example 2.2.1. Trivially, we may simply take $E \neq \{0\}$, $\boldsymbol{m}$ to be nonzero, and ψ to be the zero map of F to $\{0\} \subset L(E,G)$. Then there is an $A \in \Sigma$ for which $\boldsymbol{m}(A) \neq 0$. However, for all $e \in E$ and $B \in \Sigma$, we have that $\psi(\boldsymbol{m}(B))e = 0$, and hence $\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) = 0$.

Example 2.2.2. Let M be a closed subspace of a Hilbert space $\mathfrak{H}$ over the field $\mathbb{R}$, so that $\mathfrak{H} = M \oplus M^{\perp}$. Let $\mu : \Sigma \to M \subset H$ be σ-additive on the σ-algebra Σ and not identically zero. We have that H is continuously embedded in $L(M^{\perp}, \mathbb{R})$ under ψ, defined by

$$\psi(h)(m') \;=\; <m', h> \quad \text{for all} \;\; h \in H,\; m' \in M^{\perp}.$$

However, there exists some $A \in \Sigma$ for which $\mu(A) \neq 0$. Then, for all $m' \in M^{\perp}$ and all $B \in \Sigma$ for which $B \subset A$, we have

$$\psi(\mu(B))m' \;=\; <m', \mu(B)> \;=\; 0 \;\text{ since }\; \mu(B) \in M \;.$$

Hence, $\widetilde{\mu}_{\psi,E,G}(A) = 0$.

Example 2.2.3. Let $(\Omega, \Sigma, \mathcal{P})$ be a probability space. Let $\boldsymbol{m} : \Sigma \to L^\infty(\mathcal{P})$ by $\boldsymbol{m}(A) = 1_A$. Then $\boldsymbol{m}$ is finitely additive. Obtain a $B \in \Sigma$ for which $0 < \mathcal{P}(B) < 1$, and define K to be the closed subspace of $L^1(\mathcal{P})$:

$$K \;=\; \{\, f \in L^1(\mathcal{P}) : \mathrm{supp}(f) \subset \Omega - B \;\; a.e. \,\} \, .$$

Then K is continuously embedded in $L(L^\infty(\mathcal{P}), \mathbb{C})$ under ψ, defined by

$$\psi(f)(\ell) = \int \ell f \; \mathrm{d}\mathcal{P} \;\text{ for }\; \ell \in L^\infty \;\text{ and }\; f \in K \, .$$

Now $\boldsymbol{m}(B) = 1_B \neq 0$ in $L^\infty(\mathcal{P})$. But for all $f \in K$ and $A \in \Sigma$ for which $A \subset B$, we have $\psi(\boldsymbol{m}(A))(f) = 0$. Hence, $\widetilde{\boldsymbol{m}}_{\psi,E,G}(B) = 0$.

There do exist certain situations for which (2.3) holds, even in the nonisometric case. First, note that if ψ has a continuous linear inverse $\psi^{-1} : \psi(F) \to F$, then we have

$$\| \boldsymbol{m}(A) \|_F \;\leq\; \| \psi^{-1} \|_{L(\psi(F),F)} \, \| \psi(\boldsymbol{m}(A)) \|_{L(E,G)}$$

for $A \in \Sigma$, which would at least ensure that $\| \boldsymbol{m}(A) \|_F \leq M \, \widetilde{\boldsymbol{m}}_{\psi,E,G}(A)$ for $A \in \Sigma$, where M is a constant independent of A. Secondly, note that we have

$$\boldsymbol{m} : \Sigma \xrightarrow{\boldsymbol{m}} F \xrightarrow{\psi} \psi(F) \;\subset\; \overline{\psi(F)} \;\subset\; L(E,G) \, .$$

The set $\psi(F)$ is a linear space. If $\psi(F)$ is closed in $L(E,G)$, then it is a Banach space. Thus, if ψ is injective, continuous, linear, and $\psi(F)$ is closed, then the open mapping principle guarantees that ψ is open, and hence ψ^{-1} is continuous and linear.

Regardless of whether ψ is injective, we have the commutative diagram:

$$\begin{tikzcd}
\Sigma \arrow[r, "\boldsymbol{m}"] \arrow[dr, "\boldsymbol{m}_\theta"'] & F \arrow[r, "\psi"] \arrow[d, "\theta"] & \psi(F) & \subset & L(E,G) \\
& F/\mathrm{Ker}(\psi) \arrow[ur, "\psi_\theta"'] & & &
\end{tikzcd}$$

Here, θ is the quotient map, ψ_θ is the injective map known to make the diagram commutative, and $\boldsymbol{m}_\theta : \Sigma \to F/\mathrm{Ker}(\psi)$ is defined by $\boldsymbol{m}_\theta(A) = [\boldsymbol{m}(A)]_\theta = \theta(\boldsymbol{m}(A))$ for $A \in \Sigma$. The set function $\boldsymbol{m}_\theta$ is additive (σ-additive) when $\boldsymbol{m}$ is additive (σ-additive) due to the fact that θ is a continuous linear map. Thus, we may consider the measure

$\boldsymbol{m}_\theta : \Sigma \to F/\mathrm{Ker}(\psi)$ and the continuous linear injection $\psi_\theta : F/\mathrm{Ker}(\psi) \to L(E,G)$. Note that if $\psi(F) = \psi_\theta(F/K)$ is closed, where $K = \mathrm{Ker}(\psi)$, then we may apply the above argument to conclude that ψ_θ is open (in fact, $\psi = \psi_\theta\theta$) is open), and

$$\| \boldsymbol{m}_\theta(A) \|_{F/K} \quad \leq \quad \| \psi_\theta^{-1} \|_{L(\psi_\theta(F/K),F/K)} \| \psi_\theta(\boldsymbol{m}_\theta(A)) \|_{L(E,G)} \; .$$

But, from the diagram, we have

$$\psi_\theta \boldsymbol{m}_\theta \quad = \quad \psi \boldsymbol{m} \; ,$$

so that

$$(\widetilde{\boldsymbol{m}_\theta})_{\psi_\theta,E,G} \quad = \quad \widetilde{\boldsymbol{m}}_{\psi,E,G} \; ,$$

and thus, for $A \in \Sigma$,

$$\| \boldsymbol{m}_\theta(A) \|_{F/K} \quad \leq \quad \| \psi_\theta^{-1} \|_{L(\psi_\theta(F/K),F/K)} \, \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \; .$$

Recall that

$$\| \boldsymbol{m}_\theta(A) \|_{F/K} \quad = \quad \inf_{\substack{\psi(\boldsymbol{m}(A))=\psi(f) \\ f\in F}} \| f \|_F \; .$$

Hence, if $\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) = 0$ for some $A \in \Sigma$, we obtain a much weaker condition: there exists a sequence $\{f_n\} \subset F$ such that $\psi(\boldsymbol{m}(A)) = \psi(f_n)$ for each $n \in \mathbb{N}$ and for which $\| f_n \|_F \to 0$ (hence $f_n \to 0$ in F).

On the other hand, for general continuous linear ψ, we may consider the set function $\boldsymbol{m}_\psi : \Sigma \to \overline{\psi(F)}$ given by $\boldsymbol{m}_\psi(A) = \psi(\boldsymbol{m}(A))$ for each $A \in \Sigma$. Then $\overline{\psi(F)}$ is a Banach space and $\overline{\psi(F)} \subset L(E,G)$ by inclusion, which is, of course, an isometry. By virtue of the linearity and continuity of ψ, we have that $\boldsymbol{m}_\psi$ is an additive (σ-additive) measure when $\boldsymbol{m}$ is additive (σ-additive). We may, then, calculate the usual semivariation of $\boldsymbol{m}_\psi$. We have that

$$\begin{aligned}
(\widetilde{\boldsymbol{m}_\psi})_{E,G}(A) &= \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=1}^{n} \boldsymbol{m}_\psi(A_i) e_i \|_G \\
&= \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=1}^{n} \psi(\boldsymbol{m}(A_i)) e_i \|_G \\
&= \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \\
&= (\widetilde{\boldsymbol{m}_\theta})_{\psi_\theta,E,G}(A),
\end{aligned}$$

for each $A \in \Sigma$. Thus, each of the approaches discussed above produces the original ψ-semivariation, and for each $A \in \Sigma$,

$$\psi_\theta(\boldsymbol{m}_\theta)(A) \;=\; \psi(\boldsymbol{m}(A)) \;=\; \boldsymbol{m}_\psi(A)$$

are identical elements of $L(E,G)$. Because this action, together with the semivariation, will determine both the integral and the space of integrable functions in section 3.1, we regard these viewpoints as equivalent, although each has its own benefit. For, example, using the fact that $(\widetilde{\boldsymbol{m}}_\psi)_{E,G} = \widetilde{\boldsymbol{m}}_{\psi,E,G}$, the following properties are immediate from the properties of the usual semivariation $(\widetilde{\boldsymbol{m}}_\psi)_{E,G}$.

(1) $0 \leq \widetilde{\boldsymbol{m}}_{\psi,E,G} \leq \infty$.

(2) $\widetilde{\boldsymbol{m}}_{\psi,E,G}(\emptyset) = 0$.

(3) $\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \leq \widetilde{\boldsymbol{m}}_{\psi,E,G}(B)$ whenever $A \subset B$.

(4) $\widetilde{\boldsymbol{m}}_{\psi,E,G}$ is σ-subadditive on Σ.

(5) For any $A \in \Sigma$,

$$\begin{aligned}
\|\, \psi(\boldsymbol{m}(A)) \,\|_{L(E,G)} &= \|\, \boldsymbol{m}_\psi(A) \,\|_{L(E,G)} \\
&\leq (\widetilde{\boldsymbol{m}}_\psi)_{E,G}(A) \\
&\leq |\, \boldsymbol{m}_\psi \,|(A) \\
&= \sup_{A_i \amalg A} \sum_{i=1}^{n} \|\, \boldsymbol{m}_\psi(A_i) \,\|_{L(E,G)} \\
&= \sup_{A_i \amalg A} \sum_{i=1}^{n} \|\, \psi(\boldsymbol{m}(A_i)) \,\|_{L(E,G)} \\
&\leq \|\, \psi \,\|_{L(F,L(E,G))} \sup_{A_i \amalg A} \sum_{i=1}^{n} \|\, \boldsymbol{m}(A_i) \,\|_F \\
&= \|\, \psi \,\|_{L(F,L(E,G))} \,|\, \boldsymbol{m} \,|(A) \;,
\end{aligned}$$

so that

$$\|\, \psi(\boldsymbol{m}(A)) \,\|_{L(E,G)}(A) \;\leq\; \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \;\leq\; \|\, \psi \,\|_{L(F,L(E,G))} \,|\, \boldsymbol{m} \,|(A) \;.$$

(6) For any $A \in \Sigma$,

$$\sup_{\substack{B \in \Sigma \\ B \subset A}} \|\, \psi(\boldsymbol{m}(B)) \,\|_{L(E,G)} \;=\; \sup_{\substack{B \in \Sigma \\ B \subset A}} \|\, \boldsymbol{m}_\psi(B) \,\|_{L(E,G)} \;\leq\; \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \;.$$

(7) From properties (5) and (6), we have for $A \in \Sigma$:

$$
\begin{array}{lll}
|\, \boldsymbol{m} \,|(A) = 0 & \text{implies} & \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) = 0 \; . \\
|\, \boldsymbol{m}_{\psi} \,|(A) = 0 & \text{iff} & \widetilde{\boldsymbol{m}}_{\psi,E,G}(A) = 0 \; . \\
\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) = 0 & \text{iff} & \psi(\boldsymbol{m}(B)) = 0 \;\text{ for all }\; B \in \Sigma \;\text{ such that }\; B \subset A \; .
\end{array}
$$

(8) For any $A \in \Sigma$,

$$
\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \;=\; \sup_{z \in (G')_1} |\, (\boldsymbol{m}_{\psi})_z \,|(A) \;=\; \sup_{z \in (G')_1} |\, \boldsymbol{m}_{z\psi} \,|(A) \; ,
$$

where we define $\boldsymbol{m}_{z\psi} = (\boldsymbol{m}_{\psi})_z$.

In the last equality, recall that $\boldsymbol{m}_{z\psi} = (\boldsymbol{m}_{\psi})_z : \Sigma \to E'$ is defined for $z \in (G')_1$ by

$$
(\boldsymbol{m}_{\psi})_z(A)e \;=\; z\boldsymbol{m}_{\psi}(A)e \;=\; z\psi(\boldsymbol{m}(A))e
$$

for each $e \in E$ and $A \in \Sigma$.

We now switch to the consideration of operator-valued functions. But, as is the case for the usual semivariation, we aspire to define the semivariation in this new setting based upon the measure alone, without any function in sight. Moreover, we would like to permit different choices of spaces E, G in order to examine how the semivariation is affected by varying these spaces.

We continue to let E, F, G be Banach spaces and $\boldsymbol{m} : \Sigma \to F$ an additive measure on the σ-algebra Σ of subsets of some set T. We assume the existence of a continuous linear map $\phi : E \to L(F, G)$, and define the ϕ-semivariation by

$$
\widehat{\boldsymbol{m}}_{\phi,E,G}(A) \;=\; \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \|\sum_{i=1}^{n} \phi(e_i)\boldsymbol{m}(A_i)\,\|_G \;\text{ for }\; A \subset T \; . \tag{2.4}
$$

In subsequent chapters, this semivariation will be used to define the integral and Lebesgue space for various E-valued functions.

Rather than directly proving the basic properties of the ϕ-semivariation, we instead demonstrate its close relationship to the ψ-semivariation, from which the properties will follow.

Denote $e^\phi = \phi(e)$ for each $e \in E$, and define a map ψ on F by $\psi(f) = f^\psi$, where $f^\psi : E \to G$ by $f^\psi e = e^\phi f = \phi(e)f$ for each $e \in E$ and $f \in F$. We make the following observations:

(1) For any $f \in F$, if a net $e_\alpha \to 0$ in E, then by the continuity of ϕ, we have that $e_\alpha^\phi \to 0$ in $L(F,G)$. Thus, $e_\alpha^\phi(f) \to 0$ in G, so that $f^\psi(e_\alpha) \to 0$ in G. Hence, $\psi(f) = f^\psi$ is continuous.

(2) Given any $f \in F$, any scalar α, and any elements $e, \bar{e} \in E$, the linearity of ϕ and $\phi(e)$ gives that

$$\begin{aligned} f^\psi(\alpha e + \bar{e}) &= (\alpha e + \bar{e})^\phi(f) \\ &= (\alpha e^\phi + \bar{e}^\phi)(f) \\ &= \alpha e^\phi(f) + \bar{e}^\phi(f) \\ &= \alpha f^\psi(e) + f^\psi(\bar{e})\ . \end{aligned}$$

Thus, $\psi(f)$ is linear, so that (1) implies $\psi(f) \in L(E,G)$.

(3) Given any $f, f_o \in F$ and any scalar α, we have

$$(\alpha f + f_o)^\psi(e) \quad = \quad e^\phi(\alpha f + f_o) \quad = \quad \alpha e^\phi(f) + e^\phi(f_o) \quad = \quad \alpha f^\psi(e) + (f_o)^\psi(e)$$

for each $e \in E$, where we have used the fact that $e^\phi \in L(F,G)$. Hence, the map $\psi : F \to L(E,G)$ is linear.

(4) Given any $f \in F$, we have

$$\begin{aligned} \| f^\psi \|_{L(E,G)} &= \sup_{e \in E_1} \| f^\psi(e) \|_G \\ &= \sup_{e \in E_1} \| e^\phi(f) \|_G \\ &\leq \sup_{e \in E_1} \| \phi(e) \|_{L(F,G)} \| f \|_F \\ &= \| \phi \|_{L(E,L(F,G))} \| f \|_F\ . \end{aligned}$$

Thus, $\psi : F \to L(E,G)$ is continuous since ϕ is continuous.

Therefore, we have shown that $\psi : F \to L(E,G)$ is continuous and linear, and we may calculate the ψ-semivariation of $\boldsymbol{m}$. We have by construction, for every $A \subset T$,

$$\begin{aligned}
\widetilde{\boldsymbol{m}}_{\psi,E,G}(A) \;=\; (\widetilde{\boldsymbol{m}}_{\psi})_{E,G}(A) &= \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=n}^{n} \psi(\boldsymbol{m}(A_i)) e_i \|_G \\
&= \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=1}^{n} \phi(e_i) \boldsymbol{m}(A_i) \|_G \\
&= \widehat{\boldsymbol{m}}_{\phi,E,G}(A) \ .
\end{aligned}$$

Hence, every ϕ-semivariation may be realized as a ψ-semivariation with respect to the same spaces.

We generalize further. Define $\Phi : E \times F \to G$ by

$$\Phi(e,f) = \phi(e)f = \psi(f)e \quad \text{for each} \quad e \in E \quad \text{and} \quad f \in F \ .$$

The fact that $\psi(f) \in L(E,G)$ and $\phi(e) \in L(F,G)$ for each $(e,f) \in E \times F$ gives us that Φ is a separately continuous bilinear form. But since E, F, G are Banach spaces, we have that $\Phi \in \mathscr{B}(E,F;G)$, the space of jointly continuous bilinear forms $E \times F \to G$, as shown in the text of Treves [T], for example. Moreover, we have that

$$\begin{aligned}
\| \Phi \|_{\mathscr{B}(E,F;G)} &= \sup_{\substack{f \in F_1 \\ e \in E_1}} \| \Phi(e,f) \|_G \\
&= \sup_{\substack{f \in F_1 \\ e \in E_1}} \| e^{\phi}(f) \|_G \\
&= \sup_{e \in E_1} \| e^{\phi} \|_{L(F,G)} \\
&= \| \phi \|_{L(E,L(F,G))} \ ,
\end{aligned}$$

and likewise for $\| \psi \|_{L(F,L(E,G))}$ by the symmetry involved. Hence,

$$\| \phi \|_{L(E,L(F,G))} \;=\; \| \Phi \|_{\mathscr{B}(E,F;G)} \;=\; \| \psi \|_{L(F,L(E,G))} \ .$$

Thus, every $\phi : E \to L(F,G)$ arises from some $\Phi \in \mathscr{B}(E,F;G)$. Conversely, any bilinear map $\Phi \in \mathscr{B}(E,F;G)$ yields maps $\phi : E \to L(F,G)$ and $\psi : F \to L(E,G)$ given by $\Phi(e,f) = \phi(e)f = \psi(f)e$ for each $e \in E$ and $f \in F$, and we have $\| \phi \| = \| \Phi \| = \| \psi \|$ by the above calculation, where the norms are taken in the respective spaces of operators.

More generally, then, one might define the Φ-semivariation as

$$\widehat{M}_{\Phi,E,G}(A) \;=\; \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=1}^{n} \Phi(e_i, m(A_i)) \|_G \quad \text{for} \quad A \subset T \;,$$

given a continuous, bilinear map $\Phi : E \times F \to G$. Certainly the induced maps ϕ and ψ could independently be isometries or nonisometries. But in any case,

$$\widehat{M}_{\Phi,E,G} \;=\; \widetilde{m}_{\psi,E,G} \;=\; \widehat{m}_{\phi,E,G} \;.$$

The following properties of $\widehat{m}_{\phi,E,G}$ now follow immediately from properties (1)–(8) for $\widetilde{m}_{\psi,E,G}$ listed above, by taking ψ to be the map constructed so that $\psi(f)e = \phi(e)f$ for $e \in E$ and $f \in F$, $\widetilde{m}_{\psi,E,G} = \widehat{m}_{\phi,E,G}$, and $\| \phi \| = \| \psi \|$ in the respective spaces of operators:

(1) $0 \le \widehat{m}_{\phi,E,G} \le \infty$.

(2) $\widehat{m}_{\phi,E,G}(\emptyset) = 0$.

(3) $\widehat{m}_{\phi,E,G}(A) \le \widehat{m}_{\phi,E,G}(B)$ whenever $A \subset B$.

(4) $\widehat{m}_{\phi,E,G}$ is σ-subadditive on Σ.

(5) For any $A \in \Sigma$,

$$\begin{aligned} \| \psi(m(A)) \|_{L(E,G)} &\le \widehat{m}_{\phi,E,G}(A) \\ &\le \| \phi \|_{L(E,L(F,G))} \, | m |(A) \;. \end{aligned}$$

(6) For any $A \in \Sigma$,

$$\sup_{\substack{B \in \Sigma \\ B \subset A}} \| \psi(m(B)) \|_{L(E,G)} \;\le\; \widehat{m}_{\phi,E,G}(A) \;.$$

(7) From properties (5) and (6), we have for $A \in \Sigma$:

$$| m |(A) = 0 \quad \text{implies} \quad \widehat{m}_{\phi,E,G}(A) = 0 \;.$$

$$\widehat{m}_{\phi,E,G}(A) = 0 \qquad \text{iff} \qquad \phi(e)m(B) = 0 \;\; \text{for all} \;\; e \in E$$

$$\text{and all} \;\; B \in \Sigma \;\; \text{for which} \;\; B \subset A.$$

(8) For any $A \in \Sigma$,

$$\widehat{m}_{\phi,E,G}(A) \;=\; \sup_{z \in G_1'} | m_{z\phi} |(A) \;,$$

where we define $m_{z\phi}(A)e = z\phi(e)m(A)$ for each $e \in E$ and $A \in \Sigma$.

Note that in (5), we still do not claim the existence of any constant K for which the inequality $\| \boldsymbol{m}(A) \|_F \leq K\widehat{\boldsymbol{m}}_{\phi,E,G}$ is satisfied. In fact, the inequality does not hold in general, even when ϕ is an isometry, and this fact is the most distinct difference between the ϕ- and ψ-semivariation. One example, similar to example 2.2.2 is provided below.

Example 2.2.4. Let M be a closed subspace of a Hilbert space $\mathfrak{H}$ over the field $\mathbb{R}$, so that $\mathfrak{H} = M \oplus M^{\perp}$. Let $\mu : \Sigma \to M \subset H$ be σ-additive on the σ-algebra Σ and not identically zero. We have that H equals H' by the canonical isometry, and thus $M^{\perp}$ is isometrically embedded in $L(H, \mathbb{R})$ via this map—call it ϕ. We have

$$\phi(m')(h) \;=\; < h, m' > \quad \text{for all} \;\; h \in H \;\; \text{and} \;\; m' \in M^{\perp} \; .$$

There exists $B \in \Sigma$ for which $\mu(B) \neq 0$. However, for all $m' \in M^{\perp}$ and all $A \in \Sigma$ for which $A \subset B$, we have

$$\phi(m')(\mu(A)) \;=\; < \mu(A), m' > \;=\; 0 \;\; \text{since} \;\; \mu(A) \in M \; .$$

Hence, $\widehat{\mu}_{\phi,E,G}(B) = 0$. Note that, in fact, the ψ corresponding to this map is the map from example 2.2.2, and thus we could have immediately said $\widehat{\mu}_{\phi,E,G}(B) = 0$.

As previously demonstrated, if the map ψ that corresponds to ϕ is an isometry, or if ψ is injective and the range $\psi(F)$ is closed, we may then obtain a constant K for which $\| \boldsymbol{m}(A) \| \leq K\widehat{\boldsymbol{m}}_{\phi,E,G}$, just as before. However, there is another test for the inequality based upon ϕ alone, which we discuss below.

Fix any $g \in G$ for which $\| g \|_G = 1$. Then for each f' in F', define $u_{f'} \in L(F, G)$ by $u_{f'}(f) = f'(f)g$. Then,

$$\| u_{f'}(f) \|_G \;=\; \| f'(f)g \|_G \;=\; | f'(f) | \, \| g \|_G \;=\; | f'(f) | \, ,$$

so that

$$\| u_{f'} \|_{L(F,G)} \;=\; \sup_{f \in F_1} \| u_{f'}(f) \|_G \;=\; \sup_{f \in F_1} | f'(f) | \;=\; \| f' \|_{F'} \, .$$

Thus, F' is isometrically embedded in $L(F, G)$ in this way. Now suppose that $\phi(E)$ contains such a copy of F' and that ϕ has a continuous linear inverse. Then there is a constant K

for which $\| e \|_E \leq K \| \phi(e) \|_{L(F,G)}$. We have

$$\begin{aligned} \| f \|_F &= \sup_{f' \in (F')_1} | f'(f) | \\ &= \sup_{u_{f'} \in L(F,G)_1} \| u_{f'}(f) \|_G \\ &\leq \sup_{\phi(e) \in L(F,G)_1} \| \phi(e)(f) \|_G \\ &\leq \sup_{e \in E_K} \| \phi(e)(f) \|_G \\ &= \sup_{e \in E_1} \| \phi(Ke)(f) \|_G \\ &= K \sup_{e \in E_1} \| \phi(e)(f) \|_G \ , \end{aligned}$$

for each $f \in F$. Hence, for each $A \in \Sigma$,

$$\| \boldsymbol{m}(A) \|_F \ \leq \ K \sup_{e \in E_1} \| \phi(e) \boldsymbol{m}(A) \|_G \ \leq \ K \widehat{\boldsymbol{m}}_{\phi,E,G}(A) \ .$$

Thus, for example, we have the inequality when ϕ is injective, $\phi(E)$ is closed, and $\phi(E)$ contains a copy of F' in the manner prescribed above. In particular, this is true when ϕ is injective and $\phi(E) = L(F,G)$.

We next make note of a theorem that is known for the case of operator-valued measures. We prove it now in this new setting, and shall call upon it numerous times in the subsequent discussion, as it will provide a useful tool for creating control measures. The proof follows that of the analogous result in the aforementioned context by Dobrakov and Brooks for the finitely additive case [Br3].

Theorem 2.2.5. *Let $\boldsymbol{m} : \mathscr{R} \to F$ be finitely additive on the ring $\mathscr{R}$ of subsets of some set T, and having finite ϕ-semivariation relative to $\phi : E \to L(F,G)$. If $G \not\supset c_o$, then the set*

$$\boldsymbol{m}_{\phi,E,G} \ := \ \{ | \boldsymbol{m}_{z\phi} | : z \in (G')_1 \}$$

is uniformly strongly additive on $\mathscr{R}$. In particular, if $\boldsymbol{m}$ is σ-additive on $\mathscr{R}$, then $\boldsymbol{m}_{\phi,E,G}$ is uniformly σ-additive on $\mathscr{R}$, and equivalently, $\boldsymbol{m}_{\phi,E,G}$ is relatively weakly compact in $ca(\mathscr{R})$.

Proof. Suppose $\boldsymbol{m}_{\phi,E,G}$ is not uniformly strongly additive on $\mathscr{R}$. Then there is a sequence of disjoint sets $A_i \in \mathscr{R}$ for which it is not the case that $| \boldsymbol{m}_{z\phi} |(A_i) \to 0$ uniformly. That is, there is a sequence $\{z_i\} \subset G'_1$, disjoint $A_i \in \mathscr{R}$, and an $\epsilon > 0$ for which $| \boldsymbol{m}_{z_i\phi} |(A_i) > \epsilon$ for each $i \in \mathbb{N}$. Since each $\boldsymbol{m}_{z\phi}$ is E'-valued, it is known ([D1],[D2]) that for each $B \in \mathscr{R}$,

$$|\boldsymbol{m}_{z\phi}|(B) \;=\; (\widetilde{\boldsymbol{m}}_{z\phi})_{E,\mathbb{F}}(B) \;=\; \sup_{\substack{B_i \amalg B \\ e_i \in E_1}} |\sum_{i=1}^{n} \boldsymbol{m}_{z\phi}(B_i)\, e_i| \;.$$

Here $\mathbb{F}$ may be $\mathbb{C}$ or $\mathbb{R}$ in accordance with the vector space field under consideration. Hence, for each $i \in \mathbb{N}$, there are disjoint sets B_j^i, elements $e_j^i \in E_1$, and a number $t_i \in \mathbb{N}$ for which

$$|\sum_{j=1}^{t_i} \boldsymbol{m}_{z_i\phi}(A_i \cap B_j^i)\, e_j^i| \;>\; \epsilon \;.$$

Thus, for each $i \in \mathbb{N}$,

$$\begin{aligned} \epsilon \;&<\; |\sum_{j=1}^{t_i} \boldsymbol{m}_{z_i\phi}(A_i \cap B_j^i)\, e_j^i| \\ &=\; |\sum_{j=1}^{t_i} z_i\phi(e_j^i)\boldsymbol{m}(A_i \cap B_j^i)| \\ &=\; |z_i\Big[\sum_{j=1}^{t_i} \phi(e_j^i)\boldsymbol{m}(A_i \cap B_j^i)\Big]| \;. \end{aligned}$$

Let

$$y_i \;=\; \sum_{j=1}^{t_i} \phi(e_j^i)\boldsymbol{m}(A_i \cap B_j^i) \;\in\; G \;.$$

Then by construction, $\| y_i \|_G = \sup_{z \in (G')_1} |z(y_i)| > \epsilon$. But for $z \in (G')_1$,

$$\begin{aligned} \sum_{i=1}^{n} |z(y_i)| \;&\leq\; \sum_{i=1}^{n}\sum_{j=1}^{t_i} |\boldsymbol{m}_{z\phi}(A_i \cap B_j^i)\, e_j^i| \\ &\leq\; \sum_{i,j} \| \boldsymbol{m}_{z\phi}(A_i \cap B_j^i) \|_{E'} \\ &\leq\; |\boldsymbol{m}_{z\phi}|(T) \;\leq\; \widehat{\boldsymbol{m}}_{\phi,E,G}(T) \;<\; \infty \;. \end{aligned}$$

Hence,

$$\sum_{i=1}^{\infty} |z(y_i)| \;<\; \infty \,,$$

and for general $z \in G'$,

$$\| z \|_{G'} \sum_{i=1}^{\infty} |\frac{z}{\| z \|_{G'}}(y_i)| \;<\; \infty \;.$$

Thus, $\sum_{i=1}^{\infty} y_i$ is weakly unconditionally convergent. We now use the fact that $G \not\supset c_o$. By a theorem of Bessaga and Pelcynski [B–P], the series $\sum_{i=1}^{\infty} y_i$ converges, contrary to the fact that $\| y_i \|_G > \epsilon$ for each $i \in \mathbb{N}$. Thus, the original assumption is false, and $\boldsymbol{m}_{\phi,E,G}$ is uniformly strongly additive. When m is σ-additive, the same is true for the elements of $m_{\phi,E,G}$. In this case, uniform σ-additivity is equivalent to uniform strong additivity, which in turn is equivalent to the relative weak compactness of $m_{\phi,E,G}$ in $ca(\mathscr{R})$, as shown in a paper of Brooks and Dinculeanu [B–D1]. □

This section now concludes with a closer examination of the ϕ-semivariation in two special settings—when one of the spaces in which the functions or measures take their values is $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and the other is a Banach space over $\mathbb{F}$. The classical integration obtained in each setting would correspond to either the Bochner integral for vector-valued functions [B] or the Bartle-Dunford-Schwartz integral for vector-valued measures [D–S].

Remark 2.2.6. Let $m : \Sigma \to F$ be a σ-additive measure, where F is a Banach space, and where Σ is a σ-algebra of subsets of some set T. We have an isometry $\phi : \mathbb{F} \to L(F, F)$ given by $\phi(\alpha)f = \alpha f$. Thus, for any $A \in \Sigma$,

$$\widehat{m}_{\phi,\mathbb{F},F}(A) \;=\; \sup_{\substack{A_i \amalg A \\ |\alpha_i| \le 1}} \Big| \sum_{i=1}^{n} \phi(\alpha_i) m(A_i) \Big| \;=\; \sup_{\substack{A_i \amalg A \\ |\alpha_i| \le 1}} \Big| \sum_{i=1}^{n} \alpha_i m(A_i) \Big| \, .$$

This is exactly the definition of semivariation in the sense of Bartle-Dunford-Schwartz integral, which Brooks and Dinculeanu use to construct the Lebesgue space [B–D2].

Remark 2.2.7. Let $m : \Sigma \to \mathbb{F}$ be a σ-additive measure, where Σ is a σ-algebra of subsets of some set T. We have an isometry onto, $\phi : E \to L(\mathbb{F}, E)$, given by $\phi(e)\alpha = \alpha e$, where E is a Banach space. Then, for any $A \in \Sigma$,

$$\begin{aligned}
\widehat{m}_{\phi,E,E}(A) &= \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \Big\| \sum_{i=1}^{n} \phi(e_i) m(A_i) \Big\|_E \\
&= \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \Big\| \sum_{i=1}^{n} m(A_i) e_i \Big\|_E \\
&\le \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \sum_{i=1}^{n} | m(A_i) | \, \| e_i \|_E \\
&\le \sup_{A_i \amalg A} \sum_{i=1}^{n} | m(A_i) | \\
&= | m |(A) \, .
\end{aligned}$$

On the other hand, given a finite, pairwise disjoint subpartition $\{A_i\}_{i=1}^n$ of the set $A \in \Sigma$, we suppose that

$$\boldsymbol{m}(A_k) \;=\; a_k + b_k i \;\text{ where }\; a_k, b_k \in \mathbb{R}\;.$$

Define α_k to be 0 if $\boldsymbol{m}(A_k) = 0$, and otherwise

$$\alpha_k \;=\; \frac{a_k - b_k i}{\sqrt{a_k^2 + b_k^2}} \;=\; \frac{\overline{\boldsymbol{m}(A_k)}}{|\,\boldsymbol{m}(A_k)\,|}\;;$$

thus, $\alpha_k \boldsymbol{m}(A_k) = |\,\boldsymbol{m}(A_k)\,|$ and $|\,\alpha_k\,| \le 1$ for $k = 1, 2, \ldots, n$. Now fix any $e \in E$ for which $\|\,e\,\|_E = 1$. Then for any $k = 1, 2, \ldots, n$ we have $\|\,\alpha_k e\,\|_E \le 1$, so that

$$\begin{aligned}
\sum_{k=1}^n |\,\boldsymbol{m}(A_k)\,| \;&=\; \|\Big[\sum_{k=1}^n |\,\boldsymbol{m}(A_k)\,|\Big] e\,\|_E \\
&=\; \|\sum_{k=1}^n |\,\boldsymbol{m}(A_k)\,| e\,\|_E \\
&=\; \|\sum_{k=1}^n \alpha_k \boldsymbol{m}(A_k) e\,\|_E \\
&=\; \|\sum_{k=1}^n \phi(\alpha_k e) \boldsymbol{m}(A_k)\,\|_E \\
&\le\; \widehat{\boldsymbol{m}}_{\phi,E,E}(A)\;.
\end{aligned}$$

Hence, $|\,\boldsymbol{m}\,|(A) \le \widehat{\boldsymbol{m}}_{\phi,E,E}(A)$, and we have equality. Note then, that $|\,\boldsymbol{m}\,|$-negligible and $\boldsymbol{m}_{\phi,E,E}$-negligible sets coincide, as do the properties of finiteness for these set functions.

We have shown, then, that the ϕ-semivariation coincides with the total variation, the same measure by which the Bochner integral and the space of Bochner integrable functions is constructed, provided the said total variation is finite. Note that the seminorm for the set of Bochner integrable functions H

$$N(H) \;=\; \int \|\,H\,\|_E \,\mathrm{d}|\,\boldsymbol{m}\,|$$

is the same as that given in section 3.1. For, given $z \in E'$ and $A \in \Sigma$, we have

$$\|\,\boldsymbol{m}_{z\phi}(A)\,\|_{E'} \;=\; \sup_{e \in E_1} |\,\boldsymbol{m}_{z\phi}(A) e\,| \;=\; \sup_{e \in E_1} |\,\boldsymbol{m}(A)(ze)\,| \;=\; |\,\boldsymbol{m}(A)\,|\,\|\,z\,\|_{E'}\;,$$

so that

$$\begin{aligned} |\,\boldsymbol{m}_{z\phi}\,|(A) &= \sup_{A_i \Pi A} \sum_{i=1}^{n} \|\,\boldsymbol{m}_{z\phi}(A_i)\,\|_{E'} = \|\,z\,\|_{E'} \sup_{A_i \Pi A} \sum_{i=1}^{n} |\,\boldsymbol{m}(A_i)\,| \\ &= \|\,z\,\|_{E'}\,|\,\boldsymbol{m}\,|(A) \,, \end{aligned}$$

and hence

$$\begin{aligned} \sup_{z\in(E')_1} \int \|\,H\,\|_E \,\mathrm{d}|\,\boldsymbol{m}_z\,| &= \sup_{z\in(E')_1} \|\,z\,\|_{E'} \int \|\,H\,\|_E \,\mathrm{d}|\,\boldsymbol{m}\,| \\ &= \int \|\,H\,\|_E \,\mathrm{d}|\,\boldsymbol{m}\,| \,. \end{aligned}$$

CHAPTER 3
BANACH INTEGRATION AND THE STOCHASTIC INTEGRAL

This chapter is comprised of the construction of the Banach integration and Hilbert stochastic integration for the setting of operator-valued functions, that is E-valued functions under the influence of a continuous linear map $\phi : E \to L(F, G)$, where E, F, G are Banach spaces, and an F-valued measure $\boldsymbol{m}$. The approach here differs from that of Brooks and Dinculeanu [B–D2] for operator-valued measures, in that the integral and Lebesgue space are constructed with the use of determining sequences, more typical of the classical setting, rather than defining the integral as an element of G''. Additionally, it is not required here that ϕ satisfy $\| \phi(e) f \|_G \leq \| e \|_E \| f \|_F$, and the definition of negligible sets incorporates ϕ as to eliminate the measurability errors outlined in section 2.2. Throughout the chapter, for classical results of the theory of integration, one may refer to a standard text on integration, such as those of Halmos [H] or Dunford and Schwartz [D–S].

3.1 Banach Space Integration with Respect to ϕ-Semivariation

Throughout this section, we let E, F, G be Banach spaces consistently over either the real or complex number field and take Σ to be a σ-algebra of subsets of some set T. Let $\boldsymbol{m} : \Sigma \to F$ be an additive measure, and assume the existence of a continuous bilinear map $\Phi : E \times F \to G$, which induces a continuous linear map $\phi : E \to L(F, G)$. Recall, from (2.4), that the ϕ-semivariation of $\boldsymbol{m}$ is given by

$$\widehat{\boldsymbol{m}}_{\phi,E,G}(A) = \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=1}^{n} \phi(e_i) \boldsymbol{m}(A_i) \|_G \leq \infty \quad \text{for } A \in \Sigma \ ,$$

where E_1 denotes the unit ball of E and $A_i \amalg A$ denotes that $\{A_i\}$ is a finite collection of disjoint elements of Σ contained in A. We will sometimes write $\widehat{\boldsymbol{m}}_{E,G}$ or $\widehat{\boldsymbol{m}}$ in place of $\widehat{\boldsymbol{m}}_{\phi,E,G}$ when the context is clear.

Given an element $z \in G'$, define the additive measure $\boldsymbol{m}_z : \Sigma \to E'$ by

$$\boldsymbol{m}_z(A)e \ = \ z\phi(e)\boldsymbol{m}(A) \quad \text{for} \ \ e \in E \ \ \text{and} \ \ A \in \Sigma \ .$$

By property (8) of the ϕ-semivariation in section 2.2,

$$\widehat{m}_{\phi,E,G}(A) \;=\; \sup_{z\in(G')_1} |\,m_z\,|(A)\ , \tag{3.1}$$

for each $A \in \Sigma$. Put $m_{E,G} = \{\,|\,m_z\,| : z \in (G')_1\,\}$. We assume $\widehat{m}_{E,G}(A) < \infty$ for every $A \in \Sigma$, so that $m_{E,G}$ is a bounded set of positive measures. In the event that m is σ-additive, each m_z, and hence each $|\,m_z\,|$ is σ-additive for $z \in (G')_1$, which implies, by theorem 2.2.5, that $m_{E,G}$ is relatively weakly compact in $ca(\Sigma)$ whenever $G \not\supset c_o$.

As we have seen, the semivariation $\widehat{m}_{E,G}$ inherits most of the properties enjoyed by usual semivariation. One may even construct a continuous linear map $\psi : F \to L(E,G)$ for which $\widehat{m}_{\phi,E,G} = \widetilde{m}_{\psi,E,G}$. The most notable exception is that, whereas $m(A) = 0$ if and only if $\widetilde{m}_{\psi,E,G}(A) = 0$ when ψ is an isometry, the same is not generally true for the semivariaton, which was demonstrated in example 2.2.4.

We say a set $Q \subset T$ is *m-ϕ-negligible* if there is a $B \in \Sigma$ for which $Q \subset B$, and such that for all $e \in E$ and all $A \in \Sigma$ for which $A \subset B$, we have that $\phi(e)m(A) = 0$. That is, the action of E on all measurable subsets of B is zero. We say that $Q \subset T$ is *m-negligible* if there is a $B \in \Sigma$ for which $Q \subset B$ and such that $m(A) = 0$ for all $A \in \Sigma$ for which $A \subset B$. Note that a set $A \in \Sigma$ is m-ϕ-negligible if and only if $\widehat{m}_{\phi,E,G}(A) = 0$. If a set $Q \subset T$ is m-negligible, then it is m-ϕ-negligible, and hence $\widehat{m}_{\phi,E,G}(Q) = 0$. Once again, the reverse is generally false, even in the event that $Q \in \Sigma$ and ϕ is an isometry. We define almost everywhere conditions (denoted by *a.e.*) in the usual manner with respect to negligible sets.

We denote by $S_E(\Sigma)$, the set of E-valued, measurable simple functions of the form

$$H = \sum_{i=1}^{n} e_i 1_{A_i} \ \text{ where } e_i \in E \text{ and } A_i \in \Sigma.$$

We say $H : T \to E$ is *m-ϕ-measurable* if it is the m-ϕ-*a.e.* limit of functions $H_n \in S_E(\Sigma)$. The m-ϕ-measurable functions form a vector space under the usual addition and scalar multiplication of functions. Since $\widehat{m}_{E,G}$ is subadditive on Σ, the union of two m-ϕ-negligible sets is again m-ϕ-negligible, and thus the relation

$$H \sim K \ \text{ if and only if } H = K \ \ m\text{-}\phi\text{-}a.e.$$

is an equivalence relation. If H is $\boldsymbol{m}$-ϕ-measurable, then outside a set $A_o \in \Sigma$ for which $\widehat{\boldsymbol{m}}_{E,G}(A_o) = 0$, we have that H is the pointwise limit of measurable simple functions. It is then identified with the function $H1_{A_o}$, and this is the one we work with from the equivalence class. We call it a *true representative* of H.

Note that by (3.1), an $\boldsymbol{m}$-ϕ-negligible set $A \in \Sigma$ must satisfy $|\boldsymbol{m}_z|(A) = 0$ for all $z \in G'$. Therefore, any $\boldsymbol{m}$-ϕ-measurable H is classically measurable with respect to every $|\boldsymbol{m}_z|$ where $z \in G'$. For $\boldsymbol{m}$-ϕ-measurable H, then, we define

$$N(H) \quad = \quad \sup_{z\in(G')_1} \int \| H(t) \|_E \, \mathrm{d}|\boldsymbol{m}_z| \quad \leq \quad \infty \, .$$

Denote

$$\mathfrak{F} \quad = \quad \mathfrak{F}(\boldsymbol{m}_{E,G}) \quad = \quad \{ \boldsymbol{m}\text{-}\phi\text{-measurable } H : N(H) < \infty \} \, .$$

Then N is a seminorm on $\mathfrak{F}$, and $(\mathfrak{F}, N)$ is a seminormed space.

By definition, every $H = \sum_{i=1}^n e_i 1_{A_i} \in S_E(\Sigma)$ is $\boldsymbol{m}$-ϕ-measurable, and

$$\begin{aligned} \sup_{z\in G'} \int \| H \|_E \, \mathrm{d}|\boldsymbol{m}_z| \quad &= \quad \sup_{z\in G'} \sum_{i=1}^{n} \| e_i \|_E \, |\boldsymbol{m}_z|(A_i) \\ &\leq \quad \Big[\max_{i=1,2,\dots,n} \| e_i \|_E \Big] \, \widehat{\boldsymbol{m}}_{E,G}(A) \quad < \quad \infty \, , \end{aligned}$$

so that $S_E(\Sigma) \subset \mathfrak{F}(\boldsymbol{m}_{E,G})$. We let $\mathfrak{L} = \mathfrak{L}(\boldsymbol{m}_{E,G})$ denote the closure of $S_E(\Sigma)$ in $\mathfrak{F}(\boldsymbol{m}_{E,G})$; thus, the measurable simple functions are dense in $\mathfrak{L}(\boldsymbol{m}_{E,G})$, which is a linear subspace of $\mathfrak{F}$. Note that if H is $\boldsymbol{m}$-ϕ-measurable, then when H is identified with $H1_{A_o}$ for some $A_o \in \Sigma$ with $\widehat{\boldsymbol{m}}_{E,G}(A_o) = 0$, we have that the scalar-valued function $\| H \|_E$, defined by $\| H \|_E(t) = \| H(t) \|_E$ for $t \in T$, is the pointwise limit of simple measurable functions on T. In general, if H is Banach-valued and is the true representative, then the inverse images of open subsets of E under H belong to Σ. In particular, for any $\epsilon > 0$, we have $\{ \| H \|_E > \epsilon \} \in \Sigma$. We make the following definitions:

Definition 3.1.1. For $H, H_1, H_2, \dots$ $\boldsymbol{m}$-ϕ-measurable, we say:

(1) $\{H_n\}$ is mean Cauchy if $N(H_n - H_m) \overset{m,n}{\to} 0$.

(2) $H_n \to H$ in mean if $N(H_n - H) \overset{n}{\to} 0$.

(3) $\{H_n\}$ is Cauchy in $\boldsymbol{m}_{E,G}$-measure if, for each $\epsilon > 0$, we have $\widehat{\boldsymbol{m}}_{E,G} \{ \| H_n - H_m \|_E > \epsilon \} \overset{m,n}{\to} 0$.

(4) $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure if, for each $\epsilon > 0$, we have

$\widehat{\boldsymbol{m}}_{E,G}\{\, \| H_n - H \|_E > \epsilon \} \xrightarrow{n} 0.$

Definition 3.1.2. Given $H : T \to E$ $\boldsymbol{m}$-ϕ-measurable, we say H is *integrable* if there is a *determining sequence* $\{H_n\} \subset S_E(\Sigma)$ characterized by the properties:

(1) $\{H_n\}$ is mean Cauchy.

(2) $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure.

Let $\mathfrak{L}_o = \mathfrak{L}_o(\boldsymbol{m}_{E,G})$ denote the set of integrable functions. By definition, we have that $S_E(\Sigma) \subset \mathfrak{L}_o$. It is not immediately obvious that $\mathfrak{L}_o$ is a closed linear subspace of $\mathfrak{F}$. We shall prove in theorem 3.1.6, however, that $\mathfrak{L}_o = \mathfrak{L}$, and thus the integrable functions form a closed subspace of $\mathfrak{F}$ in which the space of E-valued, measurable simple functions are dense.

Given $H = \sum_{i=1}^{n} e_i 1_{A_i} \in S_E(\Sigma)$, we define

$$\int H \, d\boldsymbol{m} \;=\; \sum_{i=1}^{n} \phi(e_i)\boldsymbol{m}(A_i) \;,$$

and for each $A \in \Sigma$,

$$\int_A H \, d\boldsymbol{m} \;:=\; \int H 1_A \, d\boldsymbol{m} \;.$$

We call the additive set function $\nu : E \to G$, defined by $\nu(A) = \int_A H \, d\boldsymbol{m}$, the *indefinite integral* of H. It is well-defined, and it is σ-additive when $\boldsymbol{m}$ is σ-additive. Note that we have

$$\begin{aligned}
\| \int_A H \, d\boldsymbol{m} \|_G &= \sup_{z \in (G')_1} | z \int \sum_{i=1}^{n} e_i 1_{A_i \cap A} \, d\boldsymbol{m} | \\
&= \sup_{z \in (G')_1} | \sum_{i=1}^{n} z\phi(e_i)\boldsymbol{m}(A_i \cap A) | \\
&= \sup_{z \in (G')_1} | \sum_{i=1}^{n} \boldsymbol{m}_z(A_i \cap A) e_i | \\
&\le \sup_{z \in (G')_1} \sum_{i=1}^{n} \| e_i \|_E \, \| \boldsymbol{m}_z(A_i \cap A) \|_{E'} \;,
\end{aligned}$$

so that

$$\begin{aligned}
\| \int_A H \, \mathrm{d}\boldsymbol{m} \|_G &\leq \sup_{z\in(G')_1} \sum_{i=1}^{n} \| e_i \|_E \| \boldsymbol{m}_z(A_i \cap A) \|_{E'} \\
&\leq M \sup_{z\in(G')_1} \sum_{i=1}^{n} \| \boldsymbol{m}_z(A_i \cap A) \|_{E'} \\
&\leq M \sup_{z\in(G')_1} | \boldsymbol{m}_z |(\cup_{i=1}^{n} (A_i \cap A)) \\
&= M \, \widehat{\boldsymbol{m}}_{E,G}(\cup_{i=1}^{n} (A_i \cap A)) \\
&\leq M \, \widehat{\boldsymbol{m}}_{E,G}(A) ,
\end{aligned}$$

where $M = \max_i \| e_i \|_E$, and

$$\begin{aligned}
\| \int_A H \, \mathrm{d}\boldsymbol{m} \|_G &\leq \sup_{z\in(G')_1} \sum_{i=1}^{n} \| e_i \|_E \| \boldsymbol{m}_z(A_i \cap A) \|_{E'} \\
&\leq \sup_{z\in(G')_1} \sum_{i=1}^{n} \| e_i \|_E | \boldsymbol{m}_z |(A_i \cap A) \\
&= \sup_{z\in(G')_1} \int \| H 1_A \|_E \, \mathrm{d}| \boldsymbol{m}_z | \\
&= N(H 1_A) \; . \qquad (3.2)
\end{aligned}$$

Thus, both $\widehat{\boldsymbol{m}}$ and N dominate the indefinite integral.

We now construct the integral $\int H \, \mathrm{d}\boldsymbol{m}$ for general $H \in \mathfrak{L}_o$ through a series of propositions.

Proposition 3.1.3. *Let $\{H_n\} \subset S_E(\Sigma)$ be a mean Cauchy sequence, and let ν_n be the indefinite integral of H_n. Then the set function $\nu : \Sigma \to G$, defined by $\nu(A) = \lim \nu_n(A)$, exists for $A \in \Sigma$, and ν is σ-additive when $\boldsymbol{m}$ is σ-additive.*

Proof. Given $m, n \in \mathbb{N}$ and $A \in \Sigma$, we have for each $\epsilon > 0$,

$$\begin{aligned}
\| \nu_m(A) - \nu_n(A) \|_G &= \| \int_A (H_m - H_n) \, \mathrm{d}\boldsymbol{m} \|_G \\
&\leq N((H_m - H_n) 1_A), \text{ by (3.2)}, \\
&\leq N(H_m - H_n) \\
&\leq \epsilon
\end{aligned}$$

once m, n are large enough, since $\{H_n\}$ is mean Cauchy. Thus, $\{\nu_n(A)\}$ is Cauchy in G, which is complete. Hence, $\nu(A) = \lim_n \nu_n(A)$ exists in G for each $A \in \Sigma$. Note the above inequality is independent of the particular choice of $A \in \Sigma$. Thus, since

$$\begin{aligned} \| \nu_m(A) - \nu(A) \|_G &= \| \nu_m(A) - \lim_{n\to\infty} \nu_n(A) \|_G \\ &= \lim_{n\to\infty} \| \nu_m(A) - \nu_n(A) \|_G \ , \end{aligned}$$

the inequality shows that $\| \nu_m(A) - \nu(A) \|_G \xrightarrow{m} 0$ uniformly for $A \in \Sigma$. By the additivity of limits, ν is additive. Now assume $\boldsymbol{m}$ is σ-additive, so that each μ_n is σ-additive. Let $A = \cup_{i=1}^{\infty} A_i$, with $A_i \in \Sigma$ pairwise disjoint, and let $\epsilon > 0$. By the discussed uniformity, we may choose $N \in \mathbb{N}$ such that $\| \nu_n(B) - \nu(B) \|_G < \epsilon$ for all $B \in \Sigma$ once $n \geq N$. Then we may find $K \in \mathbb{N}$ for which

$$\| \nu_N(A) - \sum_{i=1}^{k} \nu_N(A_i) \|_G < \epsilon \ \text{ for all } k \geq K \ ,$$

by virtue of the σ-additivity of ν_N. Hence, for $k \geq K$,

$$\begin{aligned} \| \nu(A) - \sum_{i=1}^{k} \nu(A_i) \|_G &\leq \| \nu(A) - \nu_N(A) \|_G + \| \nu_N(A) - \sum_{i=1}^{k} \nu_N(A_i) \|_G \\ &\quad + \| \nu_N(\cup_{i=1}^{k} A_i) - \nu(\cup_{i=1}^{k} A_i) \|_G \\ &< 3\epsilon \ . \end{aligned}$$

This shows that

$$\| \nu(A) - \sum_{i=1}^{k} \nu(A_i) \|_G \xrightarrow{k} 0 \ ,$$

so that

$$\nu(A) = \lim_{k\to\infty} \sum_{i=1}^{k} \nu(A_i) = \sum_{i=1}^{\infty} \nu(A_i) \ ,$$

independent of the arrangement of A_i. □

Proposition 3.1.4. *Let* $H \in S_E(\Sigma)$, *and let* $\{H_n\} \subset S_E(\Sigma)$. *Then,*

(1) $N(H1_{(\cdot)}) \ll \widehat{\boldsymbol{m}}_{E,G}(\cdot)$.

(2) *If* $\{H_n\}$ *is mean Cauchy, then* $N(H_n1_{(\cdot)}) \overset{\text{unif}}{\ll} \widehat{\boldsymbol{m}}_{E,G}(\cdot)$.

Proof. Let $A \in \Sigma$. Then,

$$\begin{aligned} N(H1_A) &= \sup_{z\in(G')_1} \sum_{i=1}^{n} \| e_i \|_E \, | \boldsymbol{m}_z |(A_i \cap A) \\ &\leq M \sup_{z\in(G')_1} | \boldsymbol{m}_z |(\cup_{i=1}^{n}(A_i \cap A)) \\ &\leq M \widehat{\boldsymbol{m}}_{E,G}(A) \, , \end{aligned}$$

where $M = \max_i \| e_i \|_E$, which proves (1).

In order to prove (2), assume $\{H_n\}$ is mean Cauchy. Let $\epsilon > 0$. Obtain $M \in \mathbb{N}$ for which $N(H_n - H_m) \leq \frac{\epsilon}{2}$ whenever $m, n > M$. Now by (1), for $m = 1, 2, \ldots, M$ there exist $\delta_m > 0$ such that $N(H_m 1_A) \leq \frac{\epsilon}{2}$ whenever $A \in \Sigma$ and $\widehat{\boldsymbol{m}}_{E,G}(A) < \delta_m$. Take $\delta = \min(\delta_m)$. Thus, whenever $\widehat{\boldsymbol{m}}_{E,G}(A) < \delta$, we have

$$N(H_m 1_A) \leq \frac{\epsilon}{2} \leq \epsilon \text{ for } m = 1, 2, \ldots, M$$

and

$$\begin{aligned} N(H_m 1_A) &\leq N((H_m - H_M)1_A) + N(H_M 1_A) \\ &\leq N(H_m - H_M) + N(H_M 1_A) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } m > M \, . \end{aligned}$$

□

Proposition 3.1.5. *Let $\{H_n\}, \{K_n\}$ be mean Cauchy sequences from $S_E(\Sigma)$. Suppose $H : T \to E$ is an $\boldsymbol{m}$-ϕ-measurable function such that each sequence converges to H in $\boldsymbol{m}_{E,G}$-measure. Let μ_n, ν_n be the respective indefinite integrals for H_n, K_n. Then the two set functions $\mu, \nu : \Sigma \to G$ defined by $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ and $\nu(A) = \lim_{n\to\infty} \nu_n(A)$ are identical on Σ.*

Proof. The set functions μ, ν exist, as defined, by proposition 3.1.3.

Let $\epsilon > 0$, and define

$$A_n = \{ \| H_n - K_n \|_E > \epsilon \} \in \Sigma \text{ for each } n \in \mathbb{N}.$$

Since $\{H_n\}$ and $\{K_n\}$ each converge in $\boldsymbol{m}_{E,G}$-measure to H,

$$\begin{aligned} \widehat{\boldsymbol{m}}_{E,G}(A_n) &\leq \widehat{\boldsymbol{m}}_{E,G}\Big(\{\,\|H_n - H\|_E > \tfrac{\epsilon}{2}\}\cup\{\,\|K_n - H\|_E > \tfrac{\epsilon}{2}\}\Big) \\ &\leq \widehat{\boldsymbol{m}}_{E,G}\{\,\|H_n - H\|_E > \tfrac{\epsilon}{2}\} + \widehat{\boldsymbol{m}}_{E,G}\{\,\|K_n - H\|_E > \tfrac{\epsilon}{2}\} \\ &\overset{n}{\to} 0\,. \end{aligned}$$

By proposition 3.1.4, there is a $\delta > 0$ for which the conditions $A \in \Sigma$ and $\widehat{\boldsymbol{m}}_{E,G}(A) < \delta$ imply that $N(H_n 1_A) \leq \epsilon$ and $N(K_n 1_A) \leq \epsilon$ for all $n \in \mathbb{N}$.

Now given $A \in \Sigma$, we apply (3.2) to obtain

$$\begin{aligned} \|\,\nu_n(A) - \mu_n(A)\,\|_G &\leq N((H_n - K_n)1_A) \\ &\leq N((H_n - K_n)1_{A-A_n}) + N(H_n 1_{A\cap A_n}) + N(K_n 1_{A\cap A_n}) \\ &\leq \epsilon\,\widehat{\boldsymbol{m}}_{E,G}(A - A_n) + \epsilon + \epsilon, \\ &\qquad \text{once } n \text{ is large enough so that } \widehat{\boldsymbol{m}}_{E,G}(A_n) < \delta, \\ &\leq \epsilon\,(\widehat{\boldsymbol{m}}_{E,G}(T) + 2)\,. \end{aligned}$$

Since $\widehat{\boldsymbol{m}}_{E,G}(T) < \infty$, we have $\|\,\nu_n(A) - \mu_n(A)\,\|_G \overset{n}{\to} 0$, so that

$$\nu(A) = \lim_{n\to\infty} \nu_n(A) = \lim_{n\to\infty} \mu_n(A) = \mu(A) \text{ in } G\,.$$

□

By proposition 3.1.5, we may now unambiguously define for $H \in \mathfrak{L}_o$,

$$\int_A H \,\mathrm{d}\boldsymbol{m} = \lim \int_A H_n \,\mathrm{d}\boldsymbol{m} \text{ in } G\,,$$

where $\{H_n\}$ is any determining sequence for H. The set function $\nu : \Sigma \to G$ defined by

$$\nu(A) = \int_A H \,\mathrm{d}\boldsymbol{m} \text{ for } A \in \Sigma$$

is called the *indefinite integral* of H. It is additive, and σ-additive when $\boldsymbol{m}$ is σ-additive, by proposition 3.1.3.

For any $A \in \Sigma$, we see that $H1_A$ is determined by $\{H_n 1_A\}$ if $\{H_n\}$ determines H. Thus, we have

$$\begin{aligned} \int_A H \, \mathrm{d}\boldsymbol{m} &= \lim_{n\to\infty} \int_A H_n \, \mathrm{d}\boldsymbol{m} \\ &= \lim_{n\to\infty} \int H_n 1_A \, \mathrm{d}\boldsymbol{m} \\ &= \int H 1_A \, \mathrm{d}\boldsymbol{m} \,. \end{aligned}$$

Theorem 3.1.6. $\mathfrak{L}(\boldsymbol{m}_{E,G}) = \mathfrak{L}_o(\boldsymbol{m}_{E,G})$. *Given any sequence* $\{H_n\} \subset S_E(\Sigma)$ *and any* $\boldsymbol{m}$-ϕ-*measurable* H, *the following are equivalent:*

(1) $H \in \mathfrak{L}$ *and* $H_n \to H$ *in mean.*

(2) $\{H_n\}$ *is a determining sequence for* H.

Proof. ($1 \Rightarrow 2$) Let $H \in \mathfrak{L}$. Suppose $H_n \to H$ in mean. Then $N(H_n - H) \xrightarrow{n} 0$. For each $n \in \mathbb{N}$ and each $\epsilon > 0$, let

$$A_{n\epsilon} \;=\; \{\, \| H_n - H \|_E > \epsilon \,\} \;\in\; \Sigma \,.$$

Then for each fixed $\epsilon > 0$,

$$\begin{aligned} N(H_n - H) &\geq \sup_{z\in(G')_1} \int \| (H_n - H) 1_{A_{n\epsilon}} \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \\ &> \epsilon \sup_{z\in(G')_1} |\, \boldsymbol{m}_z \,|(A_{n\epsilon}) \\ &= \epsilon\, \widehat{\boldsymbol{m}}_{E,G}(A_{n\epsilon}) \,, \end{aligned}$$

so that $\widehat{\boldsymbol{m}}_{E,G}(A_{n\epsilon}) \xrightarrow{n} 0$. That is, $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure. In addition, by applying the triangle inequality to the statement $N(H_n - H) \to 0$, we see that $\{H_n\}$ is mean Cauchy. Hence, $H \in \mathfrak{L}_o$, and $\{H_n\}$ is a determining sequence for H.

($2 \Rightarrow 1$) Let $H \in \mathfrak{L}_o$, and suppose $\{H_n\}$ determines H. Fix $z \in (G')_1$. Then, for each fixed $\epsilon > 0$,

$$\begin{aligned} |\, \boldsymbol{m}_z \,|\{\, |\, \| H_n \|_E - \| H \|_E \,| > \epsilon \,\} &\leq |\, \boldsymbol{m}_z \,|\{\, \| H_n - H \|_E > \epsilon \,\} \\ &\leq \widehat{\boldsymbol{m}}_{E,G}\{\, \| H_n - H \|_E > \epsilon \,\} \xrightarrow{n} 0 \,. \end{aligned}$$

Classically, this means $\| H_n \|_E \to \| H \|_E$ in $|\, \boldsymbol{m}_z \,|$-measure. In addition,

$$\begin{aligned} \int |\, \| H_n \|_E - \| H_m \|_E \,|\, \mathrm{d}|\, \boldsymbol{m}_z \,| &\leq \int \| H_n - H_m \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \\ &\leq N(H_n - H_m) \overset{m,n}{\to} 0 \,. \end{aligned}$$

As previously shown, any $\boldsymbol{m}$-ϕ-measurable H (and likewise $\| H \|_E$) is classically measurable with respect to every $|\, \boldsymbol{m}_z \,|$ where $z \in G'$. Hence, $\{\| H_n \|_E\}$ is a determining sequence for $\| H \|_E$ relative to $|\, \boldsymbol{m}_z \,|$, which implies that

$$\int \| H \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \;=\; \lim_{n \to \infty} \int \| H_n \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \;<\; \infty \,,$$

for each $z \in (G')_1$.

Now since $\{H_n\}$ is mean Cauchy, then for each $n \in \mathbb{N}$, we may choose N_n such that $N_1 < N_2 < N_3 < \ldots$ and $N(H_{N_n} - H_m) < \dfrac{1}{n}$ for $m \geq N_n$. For each $z \in (G')_1$,

$$\begin{aligned} \int \| H_{N_n} - H \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| &= \lim_{m \to \infty} \int \| H_{N_n} - H_m \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \\ &\leq \sup_{m \geq N_n} \int \| H_{N_n} - H_m \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \\ &\leq \sup_{m \geq N_n} N(H_{N_n} - H_m) \\ &\leq \frac{1}{n} \,. \end{aligned}$$

In the above inequality we have used the fact that $\{\, \| H_{N_n} - H_m \|_E \,\}$ is a determining sequence for $\| H_{N_n} - H \|_E$ for each fixed N_n. Taking the supremum over all $z \in (G')_1$, we see that $N(H_{N_n} - H) \leq \dfrac{1}{n}$. As this holds for each $n \in \mathbb{N}$, we have $N(H_{N_n} - H) \overset{n}{\to} 0$. Moreover, for each $z \in (G')_1$,

$$\begin{aligned} \int \| H \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| &\leq \int \| H_{N_1} - H \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| + \int \| H_{N_1} \|_E \, \mathrm{d}|\, \boldsymbol{m}_z \,| \\ &\leq N(H_{N_1} - H) + N(H_{N_1}) \\ &\leq 1 + N(H_{N_1}) \,. \end{aligned}$$

Hence, $N(H) < \infty$. Since $\{H_n\}$ is mean Cauchy and $N_n \to \infty$ as $n \to \infty$, an application of triangle inequality shows $N(H_n - H) \overset{n}{\to} 0$ as well. Thus, we have shown $H \in \mathfrak{F}$ and $H_n \to H$ in mean, which implies that $H \in \mathfrak{L}$. ☐

Remark 3.1.7. We claim that the requirement of simple functions in the statement of theorem 3.1.6 can be loosened. In fact, the proof does not use the property that the H_n are simple, but for its necessity in order to label a sequence as determining for some H. Classically, the sequence $\{H_n\}$ need not be simple in order to draw the conclusion

$$\int \| H \|_E \, d|\boldsymbol{m}_z| \;=\; \lim_{n\to\infty} \int \| H_n \|_E \, d|\boldsymbol{m}_z| \;<\; \infty ,$$

but we need only that $\{ \| H_n \|_E \}$ be $|\boldsymbol{m}_z|$-measurable, mean Cauchy, and converge in $|\boldsymbol{m}_z|$-measure to $\| H \|_E$. These properties follow simply from the fact that $\{H_n\}$ is mean Cauchy and $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure. At the closure of the proof, once we have already shown that $H \in \mathfrak{F}$ and $H_n \to H$ in mean, then, provided that $\{H_n\} \subset \mathfrak{L}$, we have $H \in \mathfrak{L}$ simply from the fact that $\mathfrak{L}$ is closed in $\mathfrak{F}$. The remainder of the proof uses inequalities independent of the requirement of simple functions. Thus we have the following equivalence for $\{H_n\} \subset \mathfrak{L}$ and H $\boldsymbol{m}$-ϕ-measurable:

(1)$'$ $H \in \mathfrak{L}$ and $H_n \to H$ in mean.

(2)$'$ $\{H_n\}$ is mean Cauchy and $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure.

We now prove the following convergence theorems.

Theorem 3.1.8. *Let $\{H_n\} \subset \mathfrak{L}$, and let $H : T \to E$ be an $\boldsymbol{m}$-ϕ-measurable function. Suppose*

(1) $H_n \to H$ *in* $\boldsymbol{m}_{E,G}$*-measure.*

(2) $\lim_{\widehat{\boldsymbol{m}}_{E,G}(\cdot)\to 0} N(H_n 1_{(\cdot)}) = 0$ *uniformly for* $n = 1, 2, 3, \ldots$

Then $H \in \mathfrak{L}$ and $H_n \to H$ in mean. Conversely, if $H_n \to H$ in $\mathfrak{L}$, then properties (1) *and* (2) *both hold.*

Proof. ($\Rightarrow$) Fix $\epsilon > 0$. By (2), there is $\delta > 0$ such that $\widehat{\boldsymbol{m}}_{E,G}(A) < \delta$ implies $N(H_n 1_A) < \epsilon$ for all $n \in \mathbb{N}$. In addition, (1) implies that $\widehat{\boldsymbol{m}}_{E,G}(T - B_{mn}) \overset{m,n}{\to} 0$ where

$$B_{mn} \;=\; \left\{ \| H_n - H_m \|_E \;\le\; \frac{\epsilon}{\widehat{\boldsymbol{m}}_{E,G}(T)} \right\} .$$

Thus, there exists M_ϵ such that $\widehat{m}_{E,G}(T - B_{mn}) < \delta$ whenever $m, n \geq M_\epsilon$. Then for $m, n \geq M_\epsilon$,

$$\begin{aligned} N(H_m - H_n) &\leq N((H_m - H_n)1_{B_{mn}}) + N(H_n 1_{T-B_{mn}}) + N(H_m 1_{T-B_{mn}}) \\ &\leq \left[\frac{\epsilon}{\widehat{m}_{E,G}(T)}\right] \widehat{m}_{E,G}(B_{mn}) + \epsilon + \epsilon \\ &\leq 3\epsilon \ . \end{aligned}$$

As $\epsilon > 0$ was arbitrary, $N(H_n - H_m) \overset{m,n}{\to} 0$. Therefore, $\{H_n\}$ is mean Cauchy, which shows that $H_n \to H$ in $\mathfrak{L}$ by remark 3.1.7.

($\Leftarrow$) By remark 3.1.7, (1) is immediate.

Fix $\epsilon > 0$. Since $H \in \mathfrak{L}$, there is a $K_o \in S_E(\Sigma)$, with finite range $R(K_o)$, for which $N(K_o - H) < \frac{\epsilon}{4}$. Let $M_o = \max_{e \in R(K_o)} \| e \|_E$. Then for any $A \in \Sigma$,

$$\begin{aligned} N(H1_A) &\leq N((H - K_o)1_A) + N(K_o 1_A) \\ &\leq N(H - K_o) + M_o \widehat{m}_{E,G}(A) \ . \end{aligned}$$

Hence,

$$N(H1_A) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \text{ once } \widehat{m}_{E,G}(A) < \frac{\epsilon}{4M_o} \ .$$

Now since

$$N((H_n - H)1_A) \leq N(H_n - H) \overset{n}{\to} 0$$

for each $A \in \Sigma$, there is $N_\epsilon \in \mathbb{N}$ for which

$$N(H_n 1_A) \leq N(H1_A) + \frac{\epsilon}{2} \text{ for all } A \in \Sigma \text{ and } n \geq N_\epsilon \ .$$

Hence, for all $n \geq N_\epsilon$

$$N(H_n 1_A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } A \in \Sigma \text{ once } \widehat{m}_{E,G}(A) < \frac{\epsilon}{4M_o} =: \delta \ .$$

Repeat the beginning of this proof of the reverse direction, replacing the function H by the functions $H_1, H_2, \ldots, H_{N_\epsilon - 1}$ to obtain $M_1, M_2, \ldots, M_{N_\epsilon - 1}$ for which

$$N(H_n 1_A) < \epsilon \text{ for } A \in \Sigma \text{ once } \widehat{m}_{E,G}(A) < \frac{\epsilon}{4M_n} =: \delta_n \ .$$

Thus, for any $A \in \Sigma$ such that $\widehat{\boldsymbol{m}}_{E,G}(A) < \min\{\delta, \delta_1, \ldots, \delta_{N_\epsilon - 1}\}$, we have $N(H_n 1_A) < \epsilon$ for every $n \in \mathbb{N}$. ☐

Theorem 3.1.9. *Let $\{H_n\} \subset \mathfrak{L}$ and $K \in \mathfrak{L}$. Let H be an $\boldsymbol{m}$-ϕ-measurable function. Suppose*

(1) *$H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure.*

(2) *$\| H_n \|_E \leq \| K \|_E$ for every $n \in \mathbb{N}$.*

Then $H \in \mathfrak{L}$ and $H_n \to H$ in mean.

Proof. For each $n \in \mathbb{N}$, we have $N(H_n 1_A) \leq N(K 1_A)$ for all $A \in \Sigma$. By taking $K_n = K$ for every $n \in \mathbb{N}$ in theorem 3.1.8, we see $N(K 1_{(\cdot)}) \ll \widehat{\boldsymbol{m}}_{E,G}(\cdot)$. Hence,

$$\lim_{\widehat{\boldsymbol{m}}_{E,G}(\cdot) \to 0} N(H_n 1_{(\cdot)}) = 0 \text{ uniformly for } n = 1, 2, 3, \ldots$$

By the forward direction of theorem 3.1.8, $H_n \to H$ in mean and $H \in \mathfrak{L}$. ☐

Theorem 3.1.10. *Suppose $\boldsymbol{m}_{E,G}$ is relatively weakly compact in $ca(\Sigma)$. Then $H \in \mathfrak{L}$ whenever $H : T \to E$ is bounded and $\boldsymbol{m}$-ϕ-measurable.*

Proof. Let λ be a control measure for $\boldsymbol{m}_{E,G}$ that satisfies $\boldsymbol{m}_{E,G} \overset{\text{unif}}{\ll} \lambda \leq \widehat{\boldsymbol{m}}_{E,G}$ ([B–D1]). Since H is $\boldsymbol{m}$-ϕ-measurable, it is the pointwise limit of measurable simple functions H_n outside an $\boldsymbol{m}$-ϕ-negligible, hence λ-negligible set $A \in \Sigma$. As H is bounded, we have that H is classically integrable with respect to λ. In fact, we may choose $H_n : T \to E$ simple with $\| H_n(t) \|_E \leq \| H(t) \|_E$ for all $t \in T$, and such that $H_n \to H$ in λ-measure, a result which is well-known ([D1],[D–S]).

Now fix $\epsilon > 0$. Given any $\alpha > 0$ there is a $\delta > 0$ such that $\lambda(A) < \delta$ and $A \in \Sigma$ imply $|\boldsymbol{m}_z|(A) < \alpha$ for all $z \in (G')_1$, since $\boldsymbol{m}_{E,G} \overset{\text{unif}}{\ll} \lambda$. Hence, $\widehat{\boldsymbol{m}}_{E,G}(A) < \alpha$ by (3.1) whenever $\lambda(A) < \delta$. In particular, since $\lambda\{\| H_n - H \|_E > \epsilon\} \xrightarrow{n} 0$, we have $\widehat{\boldsymbol{m}}_{E,G}\{\| H_n - H \|_E > \epsilon\} \xrightarrow{n} 0$. Hence, $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure.

It suffices to show $\{H_n\}$ is mean Cauchy, for then $\{H_n\}$ is a determining sequence for H. Note that there exists $M \in \mathbb{N}$ for which

$$\| H_m(t) - H_n(t) \|_E \leq 2 \| H(t) \|_E \leq M \text{ for all } t \in T \text{ and } m, n \in \mathbb{N}\,.$$

Let $\epsilon > 0$. Put $B_{mn} = \{ \| H_m - H_n \|_E > \epsilon \} \in \Sigma$. By the triangle inequality, since $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure,

$$\begin{aligned} \widehat{\boldsymbol{m}}_{E,G}(B_{mn}) &\leq \widehat{\boldsymbol{m}}_{E,G}\Big(\{ \| H_n - H \|_E > \frac{\epsilon}{2} \} \cup \{ \| H_n - H \|_E > \frac{\epsilon}{2} \} \Big) \\ &\leq \widehat{\boldsymbol{m}}_{E,G} \{ \| H_n - H \|_E > \frac{\epsilon}{2} \} + \widehat{\boldsymbol{m}}_{E,G} \{ \| H_n - H \|_E > \frac{\epsilon}{2} \} \\ &\overset{m,n}{\to} 0 \,. \end{aligned}$$

Regardless of $m, n \in \mathbb{N}$,

$$\begin{aligned} N(H_n - H_m) &\leq \sup_{z \in (G')_1} \int_{B_{mn}} \| H_n - H_m \|_E \, \mathrm{d} | \boldsymbol{m}_z | \\ &\quad + \sup_{z \in (G')_1} \int_{T - B_{mn}} \| H_n - H_m \|_E \, \mathrm{d} | \boldsymbol{m}_z | \\ &\leq M \, \widehat{\boldsymbol{m}}_{E,G}(B_{mn}) + \epsilon \, \widehat{\boldsymbol{m}}_{E,G}(T - B_{mn}) \\ &\leq \epsilon \, (M + \widehat{\boldsymbol{m}}_{E,G}(T)) \end{aligned}$$

once m, n are large enough so that $\widehat{\boldsymbol{m}}_{E,G}(B_{mn}) < \epsilon$. Thus, as $\epsilon > 0$ was arbitrary and because M is independent of ϵ, we see $N(H_n - H_m) \overset{m,n}{\to} 0$. That is, the sequence $\{H_n\}$ is mean Cauchy. □

Remark 3.1.11. In particular, by theorem 2.2.5, the bounded $\boldsymbol{m}$-ϕ-measurable functions are integrable when $\boldsymbol{m}$ is σ-additive and $G \not\supset c_o$.

We end this section with a demonstration that the spaces $\mathfrak{F}(\boldsymbol{m}_{E,G})$ and $\mathfrak{L}(\boldsymbol{m}_{E,G})$ are complete.

Theorem 3.1.12. *The spaces $\mathfrak{F}(\boldsymbol{m}_{E,G})$ and $\mathfrak{L}(\boldsymbol{m}_{E,G})$ are complete.*

Proof. Since $\mathfrak{L}(\boldsymbol{m}_{E,G})$ is by definition closed in $\mathfrak{F}(\boldsymbol{m}_{E,G})$, it suffices to show that $\mathfrak{F}(\boldsymbol{m}_{E,G})$ is complete.

Let $\{H_n\}$ be a Cauchy sequence in $\mathfrak{F}$. By definition, there is an $n_1 \in \mathbb{N}$ for which $N(H_m - H_{n_1}) < \frac{1}{2}$ when $m \geq n_1$. We then choose $n_2 > n_1$ for which $N(H_m - H_{n_2}) < \frac{1}{2^2}$ when $m \geq n_2$. Continuing in this manner, we construct a subsequence $\{H_{n_k}\}$ from $\{H_n\}$ such that

$$N(H_{n_k} - H_{n_{k-1}}) < \frac{1}{2^k} \text{ for } k = 2, 3, \ldots$$

Fix $z \in (G')_1$. Note that

$$\begin{aligned} M &:= \sum_{k=2}^{\infty} N(H_{n_k} - H_{n_{k-1}}) \\ &\leq \sum_{k=2}^{\infty} \frac{1}{2^k} = \frac{1}{2} < \infty . \end{aligned}$$

But for each $m \in \mathbb{N}$, we have

$$\begin{aligned} \int \sum_{k=2}^{m} \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| &= \sum_{k=2}^{m} \int \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \\ &\leq \sum_{k=2}^{\infty} \int \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \\ &\leq \sum_{k=2}^{\infty} N(H_{n_k} - H_{n_{k-1}}) = M . \end{aligned}$$

By Lebesgue Monotone Convergence Theorem,

$$\int \sum_{k=2}^{m} \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \xrightarrow{m} \int \sum_{k=2}^{\infty} \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| ,$$

so that

$$\int \sum_{k=2}^{\infty} \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \leq M < \infty .$$

As this has been shown for arbitrary $z \in (G')_1$, we have that

$$N\Big(\sum_{k=2}^{\infty} \| H_{n_k} - H_{n_{k-1}} \|_E \Big) \leq M < \infty ;$$

in addition, we know classically, for every such z,

$$|\boldsymbol{m}_z| \{ \sum_{k=2}^{\infty} \| H_{n_k} - H_{n_{k-1}} \|_E = \infty \} = 0 ,$$

so that $\widehat{\boldsymbol{m}}_{E,G} \{ \sum_{k=2}^{\infty} \| H_{n_k} - H_{n_{k-1}} \|_E = \infty \} = 0$. Thus, we see $\sum_{k=2}^{\infty} (H_{n_k} - H_{n_{k-1}})$ is finite $\boldsymbol{m}$-ϕ-*a.e.* Now if we define the function K to be $\sum_{k=2}^{\infty} (H_{n_k} - H_{n_{k-1}})$ when the sum is finite and 0 otherwise, we see that K is $\boldsymbol{m}$-ϕ-measurable and $N(K) < \infty$ by applying Fatou's lemma. Indeed, we have for any $z \in (G')_1$,

$$\begin{aligned}\int \| K \|_E \, \mathrm{d}|\boldsymbol{m}_z| &= \int \liminf_{m\to\infty} \| \sum_{k=2}^{m} H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \\ &\leq \liminf_{m\to\infty} \int \| \sum_{k=2}^{m} H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \\ &\leq \liminf_{m\to\infty} \sum_{k=2}^{m} \int \| H_{n_k} - H_{n_{k-1}} \|_E \, \mathrm{d}|\boldsymbol{m}_z| \\ &\leq \sum_{k=2}^{\infty} N(H_{n_k} - H_{n_{k-1}}) \ < \ \infty \, .\end{aligned}$$

Note that $\sum_{k=2}^{m} H_{n_k} - H_{n_{k-1}} = H_{n_m} - H_{n_1}$, so that

$$H \ := \ K + H_{n_1} \ = \ \lim_{m\to\infty} H_{n_m} \quad \boldsymbol{m}\text{-}\phi\text{-}a.e.$$

Thus, H is $\boldsymbol{m}$-ϕ-measurable, $N(H) < \infty$, and we have for any $m \in \mathbb{N}$,

$$\begin{aligned}N(H_{n_m} - H) &= N(\sum_{k=2}^{m} (H_{n_k} - H_{n_{k-1}}) - K) \\ &= N(\sum_{k=m+1}^{\infty} (H_{n_k} - H_{n_{k-1}})) \\ &\leq \sum_{k=m+1}^{\infty} N(H_{n_k} - H_{n_{k-1}}) \\ &\quad \text{using Fatou's lemma as above,} \\ &\leq \sum_{k=m+1}^{\infty} \frac{1}{2^k} \ = \ \frac{1}{2^m} \ \overset{m}{\to} \ 0 \, .\end{aligned}$$

As the entire sequence $\{H_n\}$ was assumed Cauchy in $\mathfrak{F}$, we see, by a typical application of triangle inequality, that $H_n \to H$ in $\mathfrak{F}$. Hence, $\mathfrak{F}$ is complete. □

3.2 Application to the Stochastic Integral

In this section we apply the results of the preceding section to the context of the stochastic integral. For full background details of the stochastic integral one may refer to Déllacherie and Meyer [D–M]. A treatment of stochastic integration in the setting of operator-valued measures may be found in the papers of Brooks and Dinculeanu [B–D3] or Brooks ([Br1],[Br4]). Throughout, all spaces under consideration are vector spaces over the real field $\mathbb{R}$.

Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space, $\{\mathscr{F}_t\}_{t\geq 0}$ a filtration satisfying the usual conditions, and $X : \mathbb{R}^+ \times \Omega \to F$ a process, where for now, F is a Banach space. Given any $t \in [0, \infty)$, we define the map $X_t : \Omega \to F$ by $X_t(\omega) = X(t, \omega)$ for all $\omega \in \Omega$. We assume the process X is such that $X_t \in L^2_F$ for each $t \in [0, \infty)$.

We denote by $\mathscr{R}$ the ring of subsets of $\mathbb{R}^+ \times \Omega$ generated by the predictable rectangles:

$$\{0\} \times A \text{ where } A \in \mathscr{F}_0$$

and

$$(s, t] \times A \text{ where } A \in \mathscr{F}_s \text{ and } 0 \leq s < t < \infty .$$

We denote by $\mathscr{P} = \sigma(\mathscr{R})$ the σ-algebra generated by $\mathscr{R}$, which is referred to as the σ-algebra of predictable sets. In the customary way, we define the additive measure $I_X : \mathscr{R} \to L^2_F$, initially on the predictable rectangles by

$$\begin{aligned} I_X(\{0\} \times A) &= 1_A X_0 \\ I_X((s,t] \times A) &= 1_A(X_t - X_s) , \end{aligned}$$

and then by extending to $\mathscr{R}$ in an additive fashion.

We wish to further extend I_X to a σ-additive measure on $\mathscr{P}$. The following theorem gives a remarkably simple necessary and sufficient condition for such an extension when $F \not\supset c_o$. The details of the proof can be found in the work of Brooks and Dinculeanu [B-D3].

Theorem 3.2.1. *If $F \not\supset c_o$, then the following are equivalent:*

(1) *$I_X : \mathscr{R} \to L^2_F$ can be extended to a σ-additive measure on $\mathscr{P}$.*

(2) *I_X is bounded on $\mathscr{R}$.*

In particular, the condition $F \not\supset c_o$ holds when F is Hilbert.

Now suppose E, G are Banach spaces, and assume the existence of a continuous bilinear map $\Phi : E \times F \to G$ that induces a continuous linear map $\phi : E \to L(F, G)$. The map ϕ, in turn, induces a continuous linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$ given by $(\overline{\phi}(e)\ell)(\omega) = \phi(e)(\ell(\omega))$ for each $e \in E$, $\ell \in L^2_F$, and $\omega \in \Omega$. We may then consider the $\overline{\phi}$-semivariation of I_X, which is defined on $\mathscr{R}$ (or on $\mathscr{P}$ if an extension exists as above) by the equation

$$(\widehat{I_X})_{\overline{\phi},E,L^2_G}(A) = \sup_{\substack{A_i \amalg A \\ e_i \in E_1}} \| \sum_{i=1}^{n} \overline{\phi}(e_i) I_X(A_i) \|_{L^2_G}$$

for each $A \in \mathscr{R}$ (or $A \in \mathscr{P}$, as the case may be).

We would like to have an extension of I_X to $\mathscr{P}$ for which I_X has finite $\overline{\phi}$-semivariation. This will be the case if we restrict F, G to Hilbert space, and choose the process X to be a square-integrable martingale, as shown by propositions 3.2.3 and 3.2.4 below. The proofs resemble those in the case of operator-valued measures.

For the remainder of the section, we take F, G to be Hilbert spaces. As mentioned above, under this assumption, neither F nor G contains a copy of c_o. Note we do not require that E be a Hilbert space, a construct which will be necessary in section 6.2 for the nuclear stochastic integral.

Definition 3.2.2. We say a martingale $M : \mathbb{R}^+ \times \Omega \to F$ is square-integrable if, for each $t \geq 0$, we have $M_t \in L^2_F$, where $M_t : \Omega \to F$ is defined for each $\omega \in \Omega$ by $M_t(\omega) = M(t, \omega)$, and if

$$\sup_{t \geq 0} \| M_t \|_{L^2_F} < \infty \, .$$

Equivalently, one may say M is a square-integrable martingale if there exists $M_\infty \in L^2_F$ for which $M_t = E(M_\infty | \mathscr{F}_t)$ for every $t \geq 0$.

Proposition 3.2.3. *Let X be an F-valued square-integrable martingale. Then,*

(1) *$I_X : \mathscr{R} \to L^2_F$ can be extended to a σ-additive measure on $\mathscr{P}$.*

(2) *If Y is another F-valued square-integrable martingale, then for any disjoint sets $A, B \in \mathscr{P}$ and for any $x, y \in E$, we have:*

$$I_X(A) \perp I_Y(B) \text{ in } L^2_F \, ;$$

$$\overline{\phi}(x) I_X(A) \perp \overline{\phi}(y) I_Y(B) \text{ in } L^2_G \, .$$

Proof. First note that if X, Y are square-integrable martingales, then for any disjoint rectangles $S = (s,t] \times A$ and $T = (u,v] \times B$ in $\mathscr{R}$,

$$\begin{aligned} < I_X((s,t] \times A) \;,\; & I_Y((u,v] \times B) >_{L^2_F} \\ &= \int < 1_A(X_t - X_s),\, 1_B(Y_v - Y_u) >_F \, \mathrm{d}\mathcal{P} \\ &= \int 1_A 1_B < X_t - X_s,\, Y_v - Y_u >_F \, \mathrm{d}\mathcal{P} \;. \end{aligned} \tag{3.3}$$

If $A \cap B = \emptyset$, then $1_A 1_B \equiv 0$, so that (3.3) $= 0$. If $(s,t] \cap (u,v] = \emptyset$, assume without loss of generality that $s < t \leq u < v$ so that

$$\begin{aligned} (3.3) &= \int E\Big(1_A 1_B < X_t - X_s,\, Y_v - Y_u >_F \Big| \mathscr{F}_u\Big) \, \mathrm{d}\mathcal{P} \\ &= \int 1_A 1_B < X_t - X_s,\, E(Y_v - Y_u | \mathscr{F}_u) >_F \, \mathrm{d}\mathcal{P} \\ &= 0 \,, \end{aligned}$$

since Y is a martingale. If S is replaced by a rectangle of the form $\{0\} \times A$ in $\mathscr{R}$, then the equations remain the same with $X_t = X_0$ and $X_s \equiv 0$. Thus, again, (3.3) $= 0$. If both rectangles are of the form $\{0\} \times A$ and $\{0\} \times B$, then we must have $A \cap B = \emptyset$. Thus, $1_A 1_B \equiv 0$, so that (3.3) $= 0$. Hence, for any disjoint rectangles $S, T \in \mathscr{R}$,

$$I_X(S) \perp I_Y(T) \;.$$

In general, elements $A, B \in \mathscr{R}$ can be written as a finite union of disjoint rectangles A_i, B_k,

$$A = \cup_{i=1}^n A_i \;\text{ and }\; B = \cup_{k=1}^m B_k \;.$$

Therefore,

$$< I_X(A),\, I_Y(B) > \;=\; \sum_{i=1}^n \sum_{k=1}^m < I_X(A_i), I_Y(B_k) > \;=\; 0 \;.$$

Hence, for all disjoint sets $A, B \in \mathscr{R}$,

$$I_X(A) \perp I_Y(B) \;.$$

Now fix $A \in \mathscr{R}$, and again write A as a finite disjoint union of predictable rectangles $\{0\} \times A_0$ and $(s_i, t_i] \times A_i$, $i = 1, \ldots, n$. Let $T = \max_i (t_i)$, and put $B = [0, T] \times \Omega \in \mathscr{R}$.

Then $B - A \in \mathscr{R}$, and

$$\begin{aligned}
\| I_X(A) \|^2_{L^2_F} &\leq \| I_X(A) \|^2_{L^2_F} + \| I_X(B-A) \|^2_{L^2_F} \\
&= \| I_X(B) \|^2_{L^2_F} \text{ by orthogonality,} \\
&= \int \| X_T - X_0 \|^2_F \, \mathrm{d}\mathcal{P} \\
&\leq \int \| X_T \|^2_F \, \mathrm{d}\mathcal{P} + \int \| X_0 \|^2_F \, \mathrm{d}\mathcal{P} \\
&\leq 2 \sup \| X_t \|^2_{L^2_F} < \infty
\end{aligned}$$

since X is square-integrable. As the set $A \in \mathscr{R}$ was arbitrarily chosen, I_X is bounded on $\mathscr{R}$, and by theorem 3.2.1, the measure I_X may be extended to a σ-additive measure on $\mathscr{P}$. This proves (1).

Now consider I_X and I_Y as extended to $\mathscr{P}$ in a σ-additive fashion. Let $A \in \mathscr{R}$. The class of $B \in \mathscr{P}$ for which $I_X(A) \perp I_Y(B - A)$ is a monotone class. It contains $\mathscr{R}$ by (3.2). Therefore, by a well-known result ([H]), the class is all of $\mathscr{P}$.

Likewise, if we now fix $B \in \mathscr{P}$, then the class of $A \in \mathscr{P}$ for which $I_X(A) \perp I_Y(B - A)$ is all of $\mathscr{P}$. Thus, for all disjoint sets $A, B \in \mathscr{P}$,

$$I_X(A) \perp I_Y(B) \ .$$

Finally, if $x, y \in E$, then since $\phi(x), \phi(y) \in L(F, G)$, we have that $\phi(x)X$ and $\phi(y)Y$ are square-integrable martingales with values in G. For any $C \in \mathscr{F}$, and $0 \leq s < t$,

$$\begin{aligned}
I_{\phi(x)X}(\,(s,t] \times C\,) &= 1_C(\,(\phi(x)X)_t - (\phi(x)X)_s\,) \\
&= 1_C(\,\overline{\phi}(x)X_t - \overline{\phi}(x)X_s\,) \\
&= \overline{\phi}(x)\,(\,1_C(X_t - X_s)\,) \\
&= \overline{\phi}(x)\,I_X(\,(s,t] \times C\,) \ .
\end{aligned}$$

The result follows similarly for rectangles $\{0\} \times C$ where $C \in \mathscr{F}$. By writing any generic element $A \in \mathscr{R}$ as a union of disjoint rectangles, the fact that $\overline{\phi}(x)$ is linear and I_X is additive shows $I_{\phi(x)X}(A) = \overline{\phi}(x) I_X(A)$.

The set of $B \in \mathscr{P}$ for which $I_{\phi(x)X}(B) = \overline{\phi}(x) I_X(B)$ (having extended $I_{\phi(x)X}$ and I_X to σ-additive measures on $\mathscr{P}$) is a monotone class containing $\mathscr{R}$, and hence is all of $\mathscr{P}$. By applying the current proof to $\phi(x)X$ and $\phi(y)Y$ in place of X and Y, we have

$$I_{\phi(x)X}(A) \perp I_{\phi(y)Y}(B) \ \text{ in } \ L^2_G \ ,$$

that is,

$$\overline{\phi}(x)I_X(A) \perp \overline{\phi}(y)I_Y(B) \ \text{ in } \ L^2_G$$

for all disjoint sets $A, B \in \mathscr{P}$. □

Proposition 3.2.4. *Let X be a square-integrable martingale.*

(1) $I_X : \mathscr{R} \to L^2_F$ *has a σ-additive extension to $\mathscr{P}$ with finite $\overline{\phi}$-semivariation* $(\widehat{I_X})_{\overline{\phi},E,L^2_G}$.

(2) *The $\overline{\phi}$-semivariation of this extension is finite, regardless of the choice of embedding $\phi : E \to L(F, G)$, since*

$$(\widehat{I_X})_{\overline{\phi},E,L^2_G}(A) \ \leq \ \| \overline{\phi} \| \, \| I_X(A) \|_{L^2_F} \ \text{ for each } \ A \in \mathscr{P} \ ,$$

where $\| \overline{\phi} \| = \| \overline{\phi} \|_{L(E, L(L^2_F, L^2_G))}$.

Proof. We may extend I_X in accordance with proposition 3.2.3. Let $A \in \mathscr{P}$ and $\{A_i\} \subset \mathscr{P}$ with $A_i \amalg A$. Then $B = A - \cup_{i=1}^n A_i \in \mathscr{P}$. Let $\{e_i\}$ be a finite family of elements from E_1. By the orthogonality properties of proposition 3.2.3,

$$\begin{aligned}
\| \sum_{i=1}^n \overline{\phi}(e_i) I_X(A_i) \|^2_{L^2_G} &= \sum_{i=1}^n \| \overline{\phi}(e_i) I_X(A_i) \|^2_{L^2_G} \\
&\leq \sum_{i=1}^n \| \overline{\phi} \|^2 \, \| e_i \|^2_E \, \| I_X(A_i) \|^2_{L^2_F} \\
&\leq \| \overline{\phi} \|^2 \sum_{i=1}^n \| I_X(A_i) \|^2_{L^2_F} \\
&\leq \| \overline{\phi} \|^2 \Big(\sum_{i=1}^n \Big(\| I_X(A_i) \|^2_{L^2_F} \Big) + \| I_X(B) \|^2_{L^2_F} \Big) \\
&= \| \overline{\phi} \|^2 \, \| \sum_{i=1}^n \Big(I_X(A_i) \Big) + I_X(B) \|^2_{L^2_F} \\
&= \| \overline{\phi} \|^2 \, \| I_X(A) \|^2_{L^2_F} \ .
\end{aligned}$$

Hence, $(\widehat{I_X})_{\overline{\phi},E,L^2_G}(A) \leq \| \overline{\phi} \| \, \| I_X(A) \|_{L^2_F}$. □

Remark 3.2.5. We make two notes concerning proposition 3.2.4. First, recall that there is no reason to expect equality to hold, even if $\overline{\phi}$ were an isometry, as discussed in section 2.2, whereas equality is achieved in the case of an operator-valued measure when the map ψ is an isometry ([B–D3]). Secondly, note that, not only is the semivariation $I_X : \mathscr{P} \to L^2_F$ finite regardless of the embedding, but the definition of I_X does not depend on the embedding. Indeed, no embeddings were discussed until after the extension theorem 3.2.1. The proof of statement (1) in proposition 3.2.3 used the extension from this theorem. Hence, we may examine the same I_X under all types of embeddings of E into $L(F, G)$ where E is a Banach space and F, G are Hilbert spaces. This will be a very important part of the construction of the nuclear stochastic integral of section 6.2.

Consider now the σ-additive extension $I_X : \mathscr{P} \to L^2_F$, which has finite $\overline{\phi}$-semivariation with respect to the map $\overline{\phi} : E \to L(L^2_F, L^2_G)$. Then, in accordance with section 3.1, we may define the space $\mathfrak{F}(m_{E,L^2_G})$ and the space of integrable functions $\mathfrak{L}(m_{E,L^2_G})$, which satisfy the theorems of that section as they pertain to this setting.

For every $H \in \mathfrak{L}(m_{E,L^2_G})$, we have that

$$\int H 1_A \, \mathrm{d}X \quad = \quad \int_A H \, \mathrm{d}X \quad \in \quad L^2_G$$

for each $A \in \mathscr{P}$. Thus, we may define the process $H \cdot X = \int H \, \mathrm{d}X : \mathbb{R}^+ \times \Omega \to G$ for an integrable process H to be

$$(H \cdot X)(t, \omega) \quad = \quad (H \cdot X)_t(\omega) \quad = \quad \Big[\int H 1_{[0,t]} \, \mathrm{d}X \Big] (\omega) \,, \tag{3.4}$$

for every $t \geq 0$ and $\omega \in \Omega$.

We now close this section by demonstrating that the process defined by (3.4) is a square-integrable martingale. We first require a useful lemma that will allow us to prove results for integrable functions using the ($\mathscr{P}$-measurable) simple processes from $S_E(\mathscr{R})$ in place of those from $S_E(\mathscr{P})$.

Lemma 3.2.6. *Given any* $e1_A \in S_E(\mathscr{P})$, *where* $e \in E$ *and* $A \in \mathscr{P}$, *there exists a sequence of sets* $\{A_n\} \subset \mathscr{R}$ *for which the sequence of functions* $\{e1_{A_n}\} \subset S_E(\mathscr{R})$ *satisfies* $e1_{A_n} \xrightarrow{n} e1_A$ *in* $\mathfrak{L}(m_{E,L^2_G})$.

Proof. The process $e1_A$ is bounded and I_X-$\overline{\phi}$-measurable. Since L^2_G is a Hilbert space and therefore $L^2_G \not\supset c_o$, we see by theorem 2.2.5 that $(I_X)_{E,L^2_G}$ is relatively weakly compact in $ca(\mathscr{P})$. Thus, we utilize the control measure λ and the other general details of the proof of theorem 3.1.10 as it pertains to the stochastic setting. Since λ is positive and σ-additive on $\mathscr{P} = \sigma(\mathscr{R})$, then classically, there is a sequence of $A_n \in \mathscr{R}$ for which $\lambda(A_n \Delta A) \overset{n}{\to} 0$. But this means first of all that

$$\begin{aligned} \int \| e1_{A_n} - e1_A \|_E \, \mathrm{d}\lambda &= \int \| e1_{A_n \Delta A} \|_E \, \mathrm{d}\lambda \\ &= \| e \|_E \, \lambda(A_n \Delta A) \overset{n}{\to} 0 \, . \end{aligned}$$

Secondly we have that, for any $\epsilon > 0$,

$$\lambda \{ \| e1_{A_n} - e1_A \|_E > \epsilon \} = \lambda \{ \| e1_{A_n \Delta A} \|_E > \epsilon \}$$

converges to 0 in n. Finally, putting $A = A_0$, note that the values $\| e1_{A_n}(t, \omega) \|$ are uniformly bounded by $\| e \|$ on $\mathbb{R}^+ \times \Omega$ for $n = 0, 1, 2, \ldots$ These three properties suffice in order to follow the remainder of the proof to the conclusion that $\{e1_{A_n}\}$ is a determining sequence for $e1_A$, and hence that $e1_{A_n} \overset{n}{\to} e1_A$ in $\mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$. □

Theorem 3.2.7. *Let* $H \in \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$. *Then the process* $H \cdot X$ *is a square-integrable martingale.*

Proof. As noted above, we have that $\int H \, \mathrm{d}X \in L^2_G$ for all $A \in \mathscr{P}$. In particular, choose $A = [0, \infty) \times \Omega$. Put

$$(H \cdot X)_\infty := \int H1_{[0,\infty)} \, \mathrm{d}X \in L^2_G \, .$$

For brevity, we denote $Y = H \cdot X$, so that for each $t \geq 0$,

$$Y_t := \int H1_{[0,t]} \, \mathrm{d}X \in L^2_G \, .$$

We claim that $Y_t = E(Y_\infty | \mathscr{F}_t)$. That is, for each $t \geq 0$ and every $A \in \mathscr{F}_t$, we must show that

$$\int_A Y_t \, \mathrm{d}\mathcal{P} = \int_A Y_\infty \, \mathrm{d}\mathcal{P} \, .$$

That is, using the additivity of the integral, we must show that, for each $t \geq 0$ and every $A \in \mathscr{F}_t$,

$$\int 1_A \int H1_{(t,\infty)}\,\mathrm{d}X\,\mathrm{d}\mathcal{P} = 0\,. \tag{3.5}$$

First consider the case in which $H = e1_{C\times B}$ where $e \in E$ and either of the following hold:

(1) $C = \{0\}$ and $B \in \mathscr{F}_0$.

(2) $C = (v, w]$ where $0 \leq v < w \leq t$ and $B \in \mathscr{F}_v$.

Then the integrand in (3.5) is identically zero, and thus the equation holds trivially.

Next, consider the case in which $H = e1_{(v,w]\times B}$, where $t < w$ and $B \in \mathscr{F}_v$. Then,

$$\begin{aligned}\int H1_{(t,\infty)}\,\mathrm{d}X &= \overline{\phi}(e)\,I_X((\max\{t,v\}, w] \times B) \\ &= 1_B\overline{\phi}(e)(X_w - X_{\max\{t,v\}})\,,\end{aligned}$$

and thus, due to the fact that $A \cap B \in \mathscr{F}_{\max\{t,v\}}$ and X is a martingale, and also since $\phi(e) \in L(F, G)$, we have

$$\begin{aligned}0 &= \phi(e)\int_{A\cap B}(X_w - X_{\max\{t,v\}})\,\mathrm{d}\mathcal{P} \\ &= \int_{A\cap B}\overline{\phi}(e)(X_w - X_{\max\{t,v\}})\,\mathrm{d}\mathcal{P} \\ &= \int 1_A \int H1_{(t,\infty)}\,\mathrm{d}X\,\mathrm{d}\mathcal{P}\,.\end{aligned}$$

Thus, (3.5) holds for every $t \geq 0$ when H is of the form $e1_B$, where $e \in E$ and B is a predictable rectangle. By the additivity of the integral, and by writing any element of $\mathscr{R}$ as a finite union of disjoint rectangles, we see that (3.5) holds for every $t \geq 0$ when $H \in S_E(\mathscr{R})$.

We now apply lemma 3.2.6. Given a process $H = e1_B$, where $e \in E$ and $B \in \mathscr{P}$, there exists a sequence $\{B_n\} \subset \mathscr{R}$ for which the simple functions $H_n := e1_{B_n} \xrightarrow{n} H$ in $\mathfrak{L}(m_{E,L^2_G})$. Thus, $\{H_n\}$ is a determining sequence for H, so that by definition of the indefinite integral,

$$\int H_n 1_{(t,\infty)}\,\mathrm{d}X \xrightarrow{n} \int H1_{(t,\infty)}\,\mathrm{d}X \quad \text{in } L^2_G\,.$$

Classically, then, we have for any $A \in \mathscr{F}_t$,

$$\int 1_A \int H_n 1_{(t,\infty)}\, \mathrm{d}X\, \mathrm{d}\mathcal{P} \xrightarrow{n} \int 1_A \int H 1_{(t,\infty)}\, \mathrm{d}X\, \mathrm{d}\mathcal{P} \quad \text{in } G\,.$$

But, the left side is 0 for every $n \in \mathbb{N}$, and thus, (3.5) holds for processes of the form $H = e1_B$, where $e \in E$ and $B \in \mathscr{P}$. Once again, by the additivity of the integrals, we see that (3.5) holds for $H \in S_E(\mathscr{P})$.

Now for general $H \in \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$, there exists, by theorem 3.1.6, a determining sequence $\{H_n\} \subset S_E(\mathscr{P})$ that converges to H in $\mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$. Applying the same reasoning as in the preceding paragraph, we have that (3.5) is satisfied by $H \in \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$. $\square$

CHAPTER 4
INTEGRATION IN LOCALLY CONVEX SPACES

In this chapter we will develop an integral and Lebesgue space, including theorems on convergence, with respect to certain locally convex spaces E, F, and G, consistently chosen to be formed over one of either the real or complex fields. The main construction of this theory of integration will be provided in section 4.2. The analysis will rely heavily on the fundamental theory of locally convex spaces and topological vector spaces. One may refer to the texts of Schaeffer [S], Treves [T], or Robertson and Robertson [R–R] for an extensive discussion of such spaces. The notation used here will agree with the standard notation used by Treves [T]. Section 4.1 entails the construction of a useful class of seminorms that reduces, in a sense to be made more precise, the case of locally convex spaces to one of certain Banach spaces associated with them. Sections 4.3 and 4.4 require an understanding of nuclear locally convex spaces, the details of which can be found in the works of either Pietsch [Pi] or Schaffer [S], and in particular, nuclear spaces that satisfy the special conditions of Ustunel [U].

We first recall some notation, and then produce one result from the theory of locally convex spaces. Let E be a complete locally convex space. Given a closed convex balanced neighborhood U of 0 in E, the space $E(U)$ is the Banach space formed by the completion of the quotient of E modulo $\mathrm{Ker}(p_U)$, where p_U is the gauge of U. We denote elements of the quotient space, prior to completion, by $[\,e\,]_{p_U}$, where $e \in E$. Given a closed convex balanced bounded subset B of E, the Banach space $E[B]$ is the span of B, subject to the Grothendieck norm determined by the gauge of B.

Now let p be a continuous seminorm on E. The set

$$U_p \quad = \quad \{\, e \in E : p(e) \leq 1 \,\}$$

is a closed convex balanced neighborhood of 0 in E, and we may consider the space $E(U_p)$. For any $[\,e\,]_p \in E(U_p)$ where $e \in E$, the definition of the space $E(U_p)$ as a quotient space (prior to completion) implies that

$$\begin{aligned} \|\,[\,e\,]_p\,\|_{E(U_p)} &= \inf_{e-\bar{e}\in \mathrm{Ker}(p)} p(\bar{e}) \\ &= p(e)\ , \end{aligned} \tag{4.1}$$

since we have

$$|\,p(e)-p(\bar{e})\,| \;\leq\; p(e-\bar{e}) \;=\; 0$$

for any $e,\bar{e}\in E$ for which $e-\bar{e}\in \mathrm{Ker}(p)$.

4.1 Reducing Seminorms

Let E and G be locally convex spaces, and let F be a Banach space. We assume that there is a continuous bilinear map $\Phi : E\times F\to G$ that induces a continuous linear map $\phi : E\to L(F,G)$, given by $\phi(e)f = \Phi(e,f)$. We denote by $\mathscr{E}$ and $\mathscr{G}$ a basis of continuous seminorms for E and G, respectively. Denote by $\overline{\mathscr{E}}$ and $\overline{\mathscr{G}}$, respectively, the set of all continuous seminorms on E and G.

By virtue of the continuity of the map Φ, we have that, for each seminorm $r\in\mathscr{G}$, there exists a (continuous) seminorm $p\in\mathscr{E}$ for which $\Phi(U_p,F_1)\subset U_r$. Define $\overline{\mathscr{E}}_r$ to be the set of all $p\in\overline{\mathscr{E}}$ for which $\Phi(U_p,F_1)\subset U_r$, and put $\mathscr{E}_r = \mathscr{E}\cap\overline{\mathscr{E}}_r$. We will show that every $p\in\overline{\mathscr{E}}_r$ yields a continuous linear map of the Banach space $E(U_p)$ into the Banach space $L(F,G(U_r))$. We will further show that if the corresponding U_p is chosen in a maximal fashion (or $p\in\overline{\mathscr{E}}_r$ minimally, if one prefers), then an isometric embedding is obtained. These maps will be vital to the creation of seminorms for the Lebesgue spaces in later sections.

Suppose we fix a seminorm $r\in\mathscr{G}$. Let $z\in U_r^\circ$, which is a subset of G'. Then the maps $z\phi(e)$, defined for each $e\in E$ by $z\phi(e) = z\circ\phi(e)$, are elements of F', since $\phi(e)\in L(F,G)$. Define the map $\rho_r : E\to\overline{\mathbb{R}}$ by

$$\rho_r(e) \;=\; \sup_{z\in U_r^\circ} \|\,z\phi(e)\,\|_{F'}\ .$$

The map ρ_r has several properties which require some proof. For clarity, each property shall be written in boldface as it is addressed.

The map ρ_r is finite and nonnegative. The fact that ρ_r is nonnegative is trivial, as it is a supremum over nonnegative numbers. Now let $e\in E$, and fix any $p\in\overline{\mathscr{E}}_r$. We have already mentioned that $\overline{\mathscr{E}}_r$ (indeed, $\mathscr{E}_r$) is nonempty. Because the neighborhoods of a locally convex space are absorbing, there is a number $\lambda > 0$ for which $e \in \lambda U_p$. Hence,

$$\begin{aligned} \rho_r(e) &= \sup_{z\in U_r^\circ} \| z\phi(e) \|_{F'} = \sup_{z\in U_r^\circ} \sup_{f\in F_1} |(z\phi(e))f| \\ &= \sup_{f\in F_1} \sup_{z\in U_r^\circ} |z(\phi(e)f)| \\ &= \lambda \sup_{f\in F_1} \sup_{z\in U_r^\circ} |z(\phi(\lambda^{-1}e)f)| \\ &\leq \lambda\,, \end{aligned} \tag{4.2}$$

where the last inequality follows from the assumption $\Phi(U_p, F_1) \subset U_r$. Thus, ρ_r is finite.

The map ρ_r is subadditive and $\rho_r(\alpha e) = |\alpha|\rho_r(e)$ for all scalars α. Since ϕ and each $z \in U_r^\circ$ are linear, we have that $z\phi(\alpha e + \bar{e}) = \alpha z\phi(e) + z\phi(\bar{e})$ for each scalar α and all $e, \bar{e} \in E$. Hence, the result is immediate from the fact that (1) the norm satisfies these properties; and (2) the supremum is subadditive and preserves multiplication by positive real numbers.

The map ρ_r is a seminorm on E. This is simply a restatement of the two preceding paragraphs.

The map ρ_r is a continuous seminorm on E, and $\rho_r \leq p$ for all $p \in \overline{\mathscr{E}}_r$. In demonstrating that ρ_r is finite, equation (4.2) actually shows that given any $e \in E$, we have $\rho_r(e) \leq \lambda$ whenever $p \in \overline{\mathscr{E}}_r$, $\lambda > 0$, and $e \in \lambda U_p$. Therefore, given any $e \in E$ and $p \in \overline{\mathscr{E}}_r$, we have that $\rho_r(p(e)^{-1}e) \leq 1$, provided $p(e) \neq 0$. That is, $\rho_r(e) \leq p(e)$ when $p(e) \neq 0$. On the other hand, if $p(e) = 0$, then $p(e) \leq \lambda$ (or equivalently, $e \in \lambda U_p$) for all $\lambda > 0$, and hence $\rho_r(e) \leq \lambda$ for all $\lambda > 0$. That is, $\rho_r(e) = 0 = p(e)$. Thus, we have shown that $\rho_r(e) \leq p(e)$ for any $e \in E$ and $p \in \overline{\mathscr{E}}_r$. Since each $p \in \overline{\mathscr{E}}_r$ is continuous, fixing just one such p proves the continuity of ρ_r.

The seminorm ρ_r is an element of $\overline{\mathscr{E}}_r$. It suffices to show that

$$r(g) = \sup_{z\in U_r^\circ} |zg| \quad \text{for every } g \in G\,. \tag{4.3}$$

For then we have

$$\begin{aligned} r(\phi(e)f) &= \sup_{z\in U_r^\circ} |\, z(\phi(e)f)\, | \\ &= \sup_{z\in U_r^\circ} |\, (z\phi(e))f\, | \\ &\leq \sup_{z\in U_r^\circ} \|\, z\phi(e)\, \|_{F'} \,\|\, f\, \|_F \\ &= \rho_r(e) \,\|\, f\, \|_F \\ &\leq 1\ , \end{aligned}$$

whenever $e \in U_{\rho_r}$ and $f \in F_1$. That is, $\Phi(U_{\rho_r}, F_1) \subset U_r$.

Let $g \in G$. Given any $z \in U_r^\circ$, we have that

$$|\, z((r(g)+\epsilon)^{-1}g)\, | \leq 1 \quad \text{for every } \epsilon > 0\ ,$$

since $(r(g)+\epsilon)^{-1}g \in U_r$, so that

$$\sup_{z\in U_r^\circ} |\, zg\, | \leq r(g) + \epsilon \quad \text{for every } \epsilon > 0\ ,$$

and thus $r(g) \geq \sup\limits_{z\in U_r^\circ} |\, zg\, |$.

To show the reverse, we recall that $U_r^{\circ\circ} = U_r$ by the bipolar theorem, since U_r is closed, convex, and balanced. Hence, if $\alpha > 0$ is such that

$$|\, z(\alpha^{-1}g)\, | \leq 1 \quad \text{for every } z \in U_r^\circ\ ,$$

then $\alpha^{-1}g \in U_r^{\circ\circ} = U_r$, so that $g \in \alpha U_r$. But, since $r(g) = \inf\limits_{g\in\alpha U_r} \alpha$, we know that $g \notin (r(g)-\epsilon)U_r$ for each $0 < \epsilon < r(g)$. Thus, there must exist some $z \in U_r^\circ$ for which $|\, z(g)\, | > r(g) - \epsilon$. Hence,

$$\sup_{z\in U_r^\circ} |\, zg\, | > r(g) - \epsilon \quad \text{for every } 0 < \epsilon < r(g)\ .$$

Letting $\epsilon \to 0$ finishes the proof of (4.3).

The seminorm ρ_r is minimal in $\overline{\mathscr{E}}_r$, in the sense that $\rho_r(e) = \inf\limits_{p\in\overline{\mathscr{E}}_r} p(e)$. We have already shown that $\rho_r \in \overline{\mathscr{E}}_r$, and $\rho_r \leq p$ for all $p \in \overline{\mathscr{E}}_r$. Thus, the equality follows from the definition of infimum.

The gauge of ρ_r is maximal, in the sense that $U_{\rho_r} = \cup_{p\in\overline{\mathscr{E}}_r} U_p$**.** Since $\rho_r \leq p$, and hence $U_p \subset U_{\rho_r}$, for all $p \in \overline{\mathscr{E}}_r$, the containment is clear. The inclusion follows from the fact that U_{ρ_r} is an element of the union.

The following identity holds for all $e \in E$**.**

$$\rho_r(e) = \sup_{f\in F_1} r(\phi(e)f) \tag{4.4}$$

This follows since

$$\begin{aligned}\rho_r(e) = \sup_{z\in U_r^\circ} \| z\phi(e) \|_{F'} &= \sup_{z\in U_r^\circ} \sup_{f\in F_1} |(z\phi(e))f| \\ &= \sup_{f\in F_1} \sup_{z\in U_r^\circ} |z(\phi(e)f)| \\ &= \sup_{f\in F_1} r(\phi(e)f) ,\end{aligned}$$

where we have used (4.3) to obtain the last equality.

This concludes the list of properties of ρ_r. We shall now construct the isometric embedding of $E(U_{\rho_r})$ into the Banach space $L(F, G(U_r))$. As previously stated, we have that, given any $p \in \overline{\mathscr{E}}_r$, we may construct a continuous linear map into. However, by choosing $p = \rho_r$, we will obtain an isometry. Once again, one may refer to the works of Treves [T] or Schaeffer [S] for the specifics of the space $E(U)$ where U is a neighborhood of 0 for the locally convex space E.

Let $p \in \overline{\mathscr{E}}_r$. We first define the mapping $\phi_{pr} : E(U_p) \to L(F, G(U_r))$ on the elements $[\,e\,]_p \in E(U_p)$ where $e \in E$, which form a dense subset of $E(U_p)$, by

$$\phi_{pr}([\,e\,]_p)f = [\,\phi(e)f\,]_r \in G(U_r) \quad \text{for every } [\,e\,]_p \in E(U_p) \text{ and } f \in F .$$

It is not immediately clear that the mapping is well-defined and into.

Suppose that $[\,e\,]_p = [\,\bar{e}\,]_p$, for some $e, \bar{e} \in E$. This would mean $e - \bar{e} \in \mathrm{Ker}(p)$, so that $e - \bar{e} \in \lambda U_p$ for every $\lambda > 0$. But $\Phi(U_p, F_1) \subset U_r$ by assumption, so that $\Phi(\lambda U_p, F_1) \subset \lambda U_r$, using the fact that Φ is bilinear. Thus, given any $f \neq 0$ in F,

$$\phi(e - \bar{e})(\,\| f \|_F^{-1} f) \in \lambda U_r \quad \text{for all } \lambda > 0 ,$$

that is,

$$r(\phi(e - \bar{e})f) \leq \lambda \| f \|_F \quad \text{for all } \lambda > 0 .$$

Hence, $r(\phi(e-\bar{e})f)=0$, which holds even when $f=0$. Therefore, $\phi(e)f-\phi(\bar{e})f \in \mathrm{Ker}(r)$, and hence $[\,\phi(e)f\,]_r = [\,\phi(\bar{e})f\,]_r$ for all $f \in F$. Thus, ϕ_p is well-defined.

The map is into: Given any $e \in E$, it follows that $\phi_{pr}([\,e\,]_p)$ is linear. For, if α is any scalar and $f, \bar{f} \in F$, the fact that $\phi(e)$ and the quotient map are linear yields that

$$\begin{aligned}
\phi_{pr}([\,e\,]_p)(f+\alpha\bar{f}) &= [\,\phi(e)(f+\alpha\bar{f})\,]_r \\
&= [\,\phi(e)f+\alpha\phi(e)\bar{f}\,]_r \\
&= [\,\phi(e)f\,]_r + \alpha[\,\phi(e)\bar{f}\,]_r \\
&= \phi_{pr}([\,e\,]_p)f + \alpha\phi_{pr}([\,e\,]_p)\bar{f}\ .
\end{aligned}$$

The continuity of $\phi_{pr}([\,e\,]_p)$ for fixed $e \in E$, follows once again from the fact that (1) U_p is absorbing, so that $e \in \lambda U_p$ for some $\lambda > 0$; and (2) $p \in \overline{\mathscr{E}}_r$, so that $\Phi(U_p, F_1) \subset U_r$. For then, given any $\epsilon > 0$, we have

$$\begin{aligned}
\|\,\phi_{pr}([\,e\,]_p)f\,\|_{G(U_r)} &= \|\,[\,\phi(e)f\,]_r\,\|_{G(U_r)} \\
&= r(\phi(e)f), \quad \text{by } (4.1), \\
&= \epsilon\lambda r(\phi(\lambda^{-1}e)(\epsilon^{-1}f)) \\
&\le \epsilon\lambda\,,
\end{aligned}$$

when $\|\,f\,\|_F \le \epsilon$.

The linearity of ϕ_{pr} is quickly verified. Let $[\,e\,]_p, [\,\bar{e}\,]_p \in E(U_p)$, and let α be any scalar. Then, for each $f \in F$,

$$\begin{aligned}
\phi_{pr}(\alpha[\,e\,]_p + [\,\bar{e}\,]_p)f &= [\,\phi(\alpha e+\bar{e})f\,]_r \\
&= [\,(\alpha\phi(e)+\phi(\bar{e}))f\,]_r \\
&= [\,\alpha\phi(e)f+\phi(\bar{e})f\,]_r \\
&= \alpha[\,\phi(e)f\,]_r + [\,\phi(\bar{e})f\,]_r \\
&= \alpha\phi_{pr}([\,e\,]_p)f + \phi_{pr}([\,\bar{e}\,]_p)f \\
&= (\alpha\phi_{pr}([\,e\,]_p) + \phi_{pr}([\,\bar{e}\,]_p))f
\end{aligned}$$

Lastly, we have for each $e \in E$,

$$\begin{aligned} \| \phi_{pr}([\,e\,]_p) \|_{L(F,G(U_r))} &= \sup_{f \in F_1} \| \phi_{pr}([\,e\,]_p) f \|_{G(U_r)} \\ &= \sup_{f \in F_1} \| [\,\phi(e) f\,]_r \|_{G(U_r)} \\ &= \sup_{f \in F_1} r(\phi(e) f), \quad \text{by } (4.1), \\ &= \rho_r(e), \quad \text{by } (4.4), \\ &\leq p(e) = \| [\,e\,]_p \|_{E(U_p)}, \quad \text{by } (4.1)\,. \end{aligned}$$

This shows that the map ϕ_{pr} is continuous, and is an isometry in and only in the special case when $p = \rho_r$.

Note that since $G(U_r)$ is complete, then so is $L(F, G(U_r))$. Therefore, we may extend ϕ_{pr} uniquely from its currently dense domain to the whole space $E(U_p)$, so that the extension is continuous, linear, and an isometry in the case that $p = \rho_r$. We denote this extension also by ϕ_{pr}.

We summarize these results in the following theorem. Since the map ρ_r has been shown to belong to $\overline{\mathscr{E}}_r \subset \overline{\mathscr{E}}$, it will be denoted now by p_r. Note that we have not made the claim that $p_r \in \mathscr{E}_r$, nor that $p_r = \inf_{p \in \mathscr{E}_r} p$, although this would be very desirable.

Theorem 4.1.1. *Let* $(E, \mathscr{E}), (G, \mathscr{G})$ *be locally convex spaces, and let* F *be a Banach space. Assume that there is a continuous bilinear map* $\Phi : E \times F \to G$*, which induces the continuous linear map* $\phi : E \to L(F, G)$ *given by* $\phi(e) f = \Phi(e, f)$*. For fixed* $r \in \mathscr{G}$*, denote by* $\overline{\mathscr{E}}_r$ *the set of* $p \in \overline{\mathscr{E}}$ *that satisfy* $\Phi(U_p, F_1) \subset U_r$*. Denote* $\mathscr{E}_r = \overline{\mathscr{E}}_r \cap \mathscr{E}$*.*

Every $p \in \mathscr{E}_r$ *determines a unique continuous linear map of* $E(U_p)$ *into* $L(F, G(U_r))$ *defined by*

$$\phi_{pr}([\,e\,]_p) f = [\,\phi(e) f\,]_r$$

for $e \in E$ *and* $f \in F$*, and then extended to the completion* $E(U_p)$*.*

For each $r \in \mathscr{G}$*, there is a unique continuous seminorm* $p_r \in \overline{\mathscr{E}}_r$*, for which the map* $\phi_r := \phi_{p_r r}$ *is a continuous linear isometry of* $E(U_{p_r})$ *into* $L(F, G(U_r))$*.*

Furthermore, we have that, for each $e \in E$*,*

$$p_r(e) = \sup_{z \in U_r^\circ} \| z\phi(e) \|_{F'} = \sup_{f \in F_1} r(\phi(e) f) = \inf_{p \in \overline{\mathscr{E}}_r} p(e)\,.$$

Definition 4.1.2. Assume the hypotheses of Theorem 4.1.1. Given $r \in \mathscr{G}$, we call p_r the *reducing seminorm for* r. We call $\mathscr{E}_r$ a *reduction set for* r, and we call $\overline{\mathscr{E}}_r$ the *complete reduction set for* r.

4.2 The Case of Measures with Values in Banach Space

Throughout this section, we let E, G be complete locally convex spaces, and let F be a Banach space. As usual, $\mathscr{E}, \mathscr{G}$ denote bases of continuous seminorms for E, G respectively, and F_1 denotes the unit ball of F. The families $\mathscr{E}, \mathscr{G}$ will remain fixed throughout the discussion. Let $\boldsymbol{m} : \mathscr{S} \to F$ be an additive measure on the σ-algebra $\mathscr{S}$ of subsets of some set T. We assume the existence of a continuous bilinear map $\Phi : E \times F \to G$ that induces a continuous linear map $\phi : E \to L(F, G)$.

For each $z \in G'$, we define a set function $\overset{\circ}{\boldsymbol{m}}_z : \mathscr{S} \to E'$ by

$$\overset{\circ}{\boldsymbol{m}}_z(A)e \;=\; z\phi(e)\boldsymbol{m}(A) \;\text{ for all }\; A \in \mathscr{S} \;\text{ and }\; e \in E\ .$$

Each $z \in G'$ lies in a multiple of $U_r{}^\circ$ for some $r \in \mathscr{G}$. Fix any such r. Then $z \in \operatorname{span}(U_r{}^\circ)$, which algebraically is $G'[U_r{}^\circ]$, which in turn is $(G(U_r))'$, where $z(\,[\,g\,]_r\,) = zg$ for any $g \in G$. On the other hand, theorem 4.1.1 gives the existence of the reducing seminorm p_r and an isometry $\phi_r : E(U_{p_r}) \to L(F, G(U_r))$. This, in a fashion analogous with section 3.1, allows us to define the measure $\boldsymbol{m}_z^r : \mathscr{S} \to (E(U_{p_r}))' \;=\; E'[U_{p_r}^\circ]$, and we may calculate the total variation $|\,\boldsymbol{m}_z^r\,|$ for this measure. We have for each $e \in E$ and $A \in \mathscr{S}$,

$$\begin{aligned} \boldsymbol{m}_z^r(A)[\,e\,]_{p_r} &= z\phi_r(\,[\,e\,]_{p_r}\,)\boldsymbol{m}(A) \\ &= z[\,\phi(e)\boldsymbol{m}(A)\,]_r \\ &= z\phi(e)\boldsymbol{m}(A) \\ &= \overset{\circ}{\boldsymbol{m}}_z(A)e\ . \end{aligned}$$

Thus, we rename $\overset{\circ}{\boldsymbol{m}}_z$ to $\boldsymbol{m}_z$ and use the symbol $|\,\boldsymbol{m}_z\,|_r$ to denote $|\,\boldsymbol{m}_z^r\,|$. We will write $\boldsymbol{m}_z(A)$, $\boldsymbol{m}_z(A)e$, and $z\phi(e)\boldsymbol{m}(A)$ in place of $\boldsymbol{m}_z^r(A)$, $\boldsymbol{m}_z^r(A)[\,e\,]_{p_r}$, and $z[\,\phi(e)\boldsymbol{m}(A)\,]_r$ in the subsequent discussion.

For each $r \in \mathscr{G}$, we define the set function $\widehat{\boldsymbol{m}}_{r,E,G}$ on $\mathscr{S}$ by

$$\widehat{\boldsymbol{m}}_{r,E,G}(A) \;=\; \sup_{z \in U_r{}^\circ} |\,\boldsymbol{m}_z\,|_r(A) \;\leq\; \infty\ .$$

Note that $\widehat{\boldsymbol{m}}_{r,E,G} = \widehat{\boldsymbol{m}}_{\phi_r,E(U_{pr}),G(U_r)}$, since $U_r{}^\circ$ is the unit ball of $G'[U_r{}^\circ]$, and thus $\widehat{\boldsymbol{m}}_{r,E,G}$ inherits the corresponding properties of the ϕ_r-semivariation. Define

$$\boldsymbol{m}_{r,E,G} \;=\; \{\,|\,\boldsymbol{m}_z\,|_r : z \in U_r{}^\circ\}\;.$$

We suppose from now on that $\widehat{\boldsymbol{m}}_{r,E,G} < \infty$ on $\mathscr{S}$ for every $r \in \mathscr{G}$. Then each $\boldsymbol{m}_{r,E,G}$ is a bounded set of positive measures. Every $|\,\boldsymbol{m}_z\,|_r$ is σ-additive when $\boldsymbol{m}$ is σ-additive. In this case, by theorem 2.2.5, if any $G(U_r) \not\supset c_o$, then $\boldsymbol{m}_{r,E,G}$ is relatively weakly compact in $ca(\mathscr{S})$.

Let $S_E(\mathscr{S})$ denote the space of measurable simple functions of the form

$$K = \sum_{i=1}^{n} e_i 1_{A_i}, \quad \text{where} \quad n \in \mathbb{N},\; e_i \in E, \;\text{and}\; A_i \in \mathscr{S}\;.$$

We say $H : T \to E$ is *measurable* if it is the pointwise limit in E of elements from $S_E(\mathscr{S})$. The measurable functions form a vector space under the usual addition and scalar multiplication of functions. If H is measurable, then $p(H)$ is measurable for each $p \in \overline{\mathscr{E}}$. Given any measurable H, we define for each $r \in \mathscr{G}$,

$$\mathfrak{N}_r(H) \;=\; \sup_{z \in U_r{}^\circ} \int p_r(H)\; \mathrm{d}|\,\boldsymbol{m}_z\,|_r \;\leq\; \infty\,,$$

where again, p_r is the reducing seminorm for r. Let

$$\mathfrak{F} \;=\; \mathfrak{F}(\boldsymbol{m}_{E,G}) \;=\; \{\,\text{measurable } H : \mathfrak{N}_r(H) < \infty \;\text{ for every }\; r \in \mathscr{G}\,\}\;.$$

Then $(\,\mathfrak{F}, \{\mathfrak{N}_r\}\,)$ forms a locally convex space.

For each $H = \sum_{i=1}^{n} e_i 1_{A_i} \in S_E(\mathscr{S})$ and every $r \in \mathscr{G}$, we have

$$\begin{aligned} \mathfrak{N}_r(H) \;&=\; \sup_{z \in U_r{}^\circ} \int p_r(H)\; \mathrm{d}|\,\boldsymbol{m}_z\,|_r \\ &\leq\; \Big[\max_{i=1,2,\ldots,n} p_r(e_i)\Big]\, \widehat{\boldsymbol{m}}_{r,E,G}(T) \;<\; \infty\,. \end{aligned} \tag{4.5}$$

Hence, $S_E(\mathscr{S}) \subset \mathfrak{F}$. Let $\mathfrak{L} = \mathfrak{L}(\boldsymbol{m}_{E,G})$ denote the closure of $S_E(\mathscr{S})$ in $\mathfrak{F}$; thus, $S_E(\mathscr{S})$ is dense in the linear subspace $\mathfrak{L} \subset \mathfrak{F}$. Note that, for H measurable, $p_r(H)$ is a pointwise limit of simple functions. Since it is scalar-valued, we have that $p_r^{-1}(\mathcal{O}) \in \mathscr{S}$ whenever $\mathcal{O}$ is open in the scalar field. We make the following definitions:

Definition 4.2.1. For $H, H_1, H_2, \ldots$ measurable, we say:

(1) $\{H_n\}$ is *mean Cauchy* if $\mathfrak{N}_r(H_n - H_m) \overset{m,n}{\to} 0$ for each $r \in \mathscr{G}$.

(2) $H_n \to H$ *in mean* if $\mathfrak{N}_r(H_n - H) \overset{n}{\to} 0$ for each $r \in \mathscr{G}$.

(3) $\{H_n\}$ is *Cauchy in $\boldsymbol{m}_{E,G}$-measure* if, for each $r \in \mathscr{G}$ and $\epsilon > 0$, we have $\widehat{\boldsymbol{m}}_{r,E,G}\{\, p_r(H_n - H_m) > \epsilon\} \overset{m,n}{\to} 0$.

(4) $H_n \to H$ *in $\boldsymbol{m}_{E,G}$-measure* if, for each $r \in \mathscr{G}$ and $\epsilon > 0$, we have $\widehat{\boldsymbol{m}}_{r,E,G}\{\, p_r(H_n - H) > \epsilon\} \overset{n}{\to} 0$.

Definition 4.2.2. Given $H : T \to E$ measurable, we say H is *integrable* if there is a *determining sequence* $\{H_n\} \subset S_E(\mathscr{S})$ defined by

(1) $\{H_n\}$ is mean Cauchy.

(2) $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure.

Denote by $\mathfrak{L}_o = \mathfrak{L}_o(\boldsymbol{m}_{E,G})$ the set of integrable functions. Note that $S_E(\mathscr{S}) \subset \mathfrak{L}_o$. It is not immediately clear that $\mathfrak{L}_o$ is a closed linear subspace of $\mathfrak{F}$. We shall prove, however, that $\mathfrak{L}_o = \mathfrak{L}$ in theorem 4.2.9, and thus, the integrable functions form a closed subspace of $\mathfrak{F}$ in which the space of E-valued, measurable simple functions are dense. The reason for calling such functions integrable will become clear as we will be able to construct integrals for them, beginning with the simple functions.

For $H = \sum_{i=1}^{n} e_i 1_{A_i} \in S_E(\mathscr{S})$ and $A \in \mathscr{S}$, we define

$$\int H \, d\boldsymbol{m} \;=\; \sum_{i=1}^{n} \phi(e_i)\boldsymbol{m}(A_i) \;\in\; G \,,$$

and

$$\int_A H \, d\boldsymbol{m} \;=\; \int H 1_A \, d\boldsymbol{m} \,.$$

The additive set function $\int_{(\cdot)} H \, d\boldsymbol{m}$ is called the *indefinite integral* for H. The indefinite integral of $H \in S_E(\mathscr{S})$ is well-defined and additive, and is σ-additive when $\boldsymbol{m}$ is σ-additive.

We shall require two lemmas in the construction of the integral for general $H \in \mathfrak{L}$.

Lemma 4.2.3. *Given $e' \in E'[U_p^\circ]$, where $p \in \overline{\mathscr{E}}$, we have for each $e \in E$,*

$$|\, e'(e) \,| \;\le\; \|\, e' \,\|_{E'[U_p^\circ]}\, p(e) \,.$$

Proof. We have that

$$\| e' \|_{E'[U_p^\circ]} \;=\; \inf_{e' \in \alpha U_p^\circ} \alpha \, .$$

For any $\epsilon, \delta > 0$,

$$e' \;\in\; (\, \| e' \|_{E'[U_p^\circ]} + \epsilon \,) \, U_p^\circ \, ,$$

so that

$$| \, e'\Big(\frac{e}{p(e) + \delta}\Big) \, | \;\leq\; \| e' \|_{E'[U_p^\circ]} + \epsilon \, .$$

Hence,

$$| \, e'(e) \, | \;\leq\; (\, \| e' \|_{E'[U_p^\circ]} + \epsilon)(p(e) + \delta) \, .$$

Letting $\epsilon, \delta \to 0$ proves the result. □

In particular, by lemma 4.2.3, we have for any $r \in \mathscr{G}$ and $z \in U_r{}^\circ$,

$$| \, \boldsymbol{m}_z(A)e \, | \;\leq\; \| \, \boldsymbol{m}_z(A) \, \|_{E'[U_{p_r}^\circ]} \, p_r(e) \tag{4.6}$$

for all $e \in E$ and $A \in \mathscr{S}$. We use this fact to prove the next lemma.

Lemma 4.2.4. *For $H \in S_E(\mathscr{S})$ and $r \in \mathscr{G}$, we have*

$$r\Big(\int_A H \, \mathrm{d}\boldsymbol{m} \Big) \;\leq\; \mathfrak{N}_r(H 1_A) \, ,$$

for each $A \in \mathscr{S}$.

Proof. Let $H = \sum_{i=1}^n e_i 1_{A_i} \in S_E(\mathscr{S})$. Then for $z \in U_r{}^\circ$, we have

$$\begin{aligned} z \int H \, \mathrm{d}\boldsymbol{m} \;&=\; z \sum_{i=1}^n \phi(e_i) \boldsymbol{m}(A_i) \\ &=\; \sum_{i=1}^n z\phi(e_i) \boldsymbol{m}(A_i) \\ &=\; \sum_{i=1}^n \boldsymbol{m}_z(A_i) e_i \, . \end{aligned}$$

Now

$$\begin{aligned} r\Big(\int H \,\mathrm{d}\boldsymbol{m}\Big) &= \sup_{z\in U_r{}^\circ} |\, z \int H \,\mathrm{d}\boldsymbol{m} \,| \quad \text{by (4.3)}, \\ &= \sup_{z\in U_r{}^\circ} |\sum_{i=1}^{n} \boldsymbol{m}_z(A_i)e_i \,| \\ &\le \sup_{z\in U_r{}^\circ} \sum_{i=1}^{n} |\, \boldsymbol{m}_z(A_i)e_i \,| \\ &\le \sup_{z\in U_r{}^\circ} \sum_{i=1}^{n} \|\, \boldsymbol{m}_z(A_i) \,\|_{E'[U^\circ_{p_r}]}\, p_r(e_i) \quad \text{by (4.6)}, \\ &\le \sup_{z\in U_r{}^\circ} \sum_{i=1}^{n} p_r(e_i) |\, \boldsymbol{m}_z \,|_r(A_i) \\ &= \sup_{z\in U_r{}^\circ} \int p_r(H) \,\mathrm{d}|\, \boldsymbol{m}_z \,|_r \\ &= \mathfrak{N}_r(H) \;. \end{aligned}$$

The lemma is proved by replacing H by $H1_A$ for $A \in \mathscr{S}$. ☐

We may now construct the integral $\int H \,\mathrm{d}\boldsymbol{m}$ for integrable H through a series of propositions.

Proposition 4.2.5. *Let* $\{H_n\} \subset S_E(\mathscr{S})$ *be a mean Cauchy sequence, and let* $\nu_n : \mathscr{S} \to G$ *denote the indefinite integral of* H_n *for each* $n \in \mathbb{N}$. *Then the set function* $\nu : \mathscr{S} \to G$, *defined by* $\nu(A) = \lim \nu_n(A)$, *exists for* $A \in \mathscr{S}$. *In addition,* ν *is* σ*-additive when* $\boldsymbol{m}$ *is* σ*-additive.*

Proof. Let $m, n \in \mathbb{N}$ and $A \in \mathscr{S}$. Fix $r \in \mathscr{G}$. Then,

$$\begin{aligned} r(\nu_m(A) - \nu_n(A)) &= r\Big(\int_A (H_m - H_n) \,\mathrm{d}\boldsymbol{m}\Big) \\ &\le \mathfrak{N}_r((H_m - H_n)1_A), \quad \text{by lemma 4.2.4}, \\ &\le \mathfrak{N}_r(H_m - H_n) \overset{m,n}{\to} 0 \;, \end{aligned}$$

since $\{H_n\}$ is mean Cauchy. As $r \in \mathscr{G}$ was arbitrary, $\{\nu_n(A)\}$ is Cauchy in G, which is assumed complete. Hence, $\nu(A) = \lim \nu_n(A)$ exists in G. Note that the above inequality is independent of the particular choice of $A \in \mathscr{S}$. Thus, since for each $r \in \mathscr{G}$,

$$\begin{aligned} r(\nu_m(A) - \nu(A)) &= r(\nu_m(A) - \lim_{n\to\infty} \nu_n(A)) \\ &= \lim_{n\to\infty} r(\nu_m(A) - \nu_n(A)) , \end{aligned}$$

we have $r(\nu_m(A) - \nu(A)) \to 0$ uniformly for $A \in \mathscr{S}$. By the additivity of limits, the set function ν is additive.

Now suppose that $\boldsymbol{m}$ is σ-additive, and let $A = \cup_{i=1}^{\infty} A_i$ for some disjoint sets $A_i \in \mathscr{S}$. We have the inequality:

$$\begin{aligned} r\Big(\nu(A) - \sum_{i=1}^{k} \nu(A_i)\Big) &\leq r\Big(\nu(A) - \nu_K(A)\Big) + r\Big(\nu_K(A) - \sum_{i=1}^{k} \nu_K(A_i)\Big) \\ &\quad + r\Big(\nu_K(\cup_{i=1}^{k} A_i) - \nu(\cup_{i=1}^{k} A_i)\Big) , \end{aligned}$$

which holds for $K \in \mathbb{N}$, $r \in \mathscr{G}$. For fixed $r \in \mathscr{G}$, the first and third terms are made small by choosing K large, independent of both k and the arrangement of the A_i, due to the mentioned uniformity. Then k may be chosen to make the middle term small, since ν_K is σ-additive. This shows that, independent of the arrangement of the sets A_i,

$$r\Big(\nu(A) - \sum_{i=1}^{k} \nu(A_i)\Big) \xrightarrow{k} 0 \text{ for each } r \in \mathscr{G} ,$$

so that

$$\nu(A) = \sum_{i=1}^{\infty} \nu(A_i) .$$

Hence ν is σ-additive. □

Definition 4.2.6. Given any set function $\boldsymbol{M} : \mathscr{S} \to X$, where X is a locally convex space, and a set function $\mu : \mathscr{S} \to \mathbb{R}^+$, we say $\boldsymbol{M}$ is absolutely continuous with respect to μ, written $\boldsymbol{M} \ll \mu$, if for every continuous seminorm r on X (or equivalently, in a basis of continuous seminorms on X), we have

$$\lim_{\mu(B)\to 0} r(\boldsymbol{M}(B)) \to 0 .$$

That is, given $\epsilon > 0$, there is $\delta > 0$ such that $r(\boldsymbol{M}(B)) < \epsilon$ whenever $B \in \mathscr{S}$ and $\mu(B) < \delta$.

A sequence of set functions $\{\boldsymbol{M}_n\}$ is said to be uniformly absolutely continuous with respect to μ, written $\boldsymbol{M}_n \overset{\text{unif}}{\ll} \mu$, if for every continuous seminorm r on X (or equivalently, in a basis of continuous seminorms on X), we have

$$\lim_{\mu(B)\to 0} r(\boldsymbol{M}_n(B)) \overset{\text{unif}}{\rightarrow} 0 \ .$$

That is, given $\epsilon > 0$, there is $\delta > 0$ such that $r(\boldsymbol{M}_n(B)) < \epsilon$ for all $n \in \mathbb{N}$, whenever $B \in \mathscr{S}$ and $\mu(B) < \delta$.

Proposition 4.2.7. *Let $\nu, \nu_1, \nu_2, \ldots$ denote the indefinite integrals of the functions $H, H_1, H_2, \ldots$ from $S_E(\mathscr{S})$. We have that*

(1) $\nu \ll \widehat{\boldsymbol{m}}_{r,E,G}$, *and* $\mathfrak{N}_r(H1_{(\cdot)}) \ll \widehat{\boldsymbol{m}}_{r,E,G}$ *for each* $r \in G$.

(2) *If* $\{H_n\}$ *is mean Cauchy, then* $\nu_n \overset{\text{unif}}{\ll} \widehat{\boldsymbol{m}}_{r,E,G}$ *and* $\mathfrak{N}_r(H1_{(\cdot)}) \overset{\text{unif}}{\ll} \widehat{\boldsymbol{m}}_{r,E,G}$ *for each* $r \in \mathscr{G}$.

Proof. Fix $r \in \mathscr{G}$. Let $A \in \mathscr{S}$. Then by (4.5) and lemma 4.2.4,

$$r(\nu(A)) \ \le \ \mathfrak{N}_r(H1_A) \ \le \ \Big[\max_{e_i \in R(H)} p_r(e_i)\Big] \, \widehat{\boldsymbol{m}}_{r,E,G}(A) \ ,$$

where $R(H) = \text{Range}(H)$. This proves (1). In order to prove (2), assume $\{H_n\}$ is mean Cauchy. Fix $r \in \mathscr{G}$, and let $\epsilon > 0$. Obtain $K \in \mathbb{N}$ for which $\mathfrak{N}_r(H_n - H_m) \le \frac{\epsilon}{2}$ whenever $m, n \ge K$. By part (1), for $m = 1, 2, \ldots, K$ there exist $\delta_m > 0$ such that $\mathfrak{N}_r(H_m 1_A) \le \frac{\epsilon}{2}$ whenever $\widehat{\boldsymbol{m}}_{r,E,G}(A) < \delta_m$ and $A \in \mathscr{S}$. Take $\delta = \min(\delta_m)$. Thus, whenever $\widehat{\boldsymbol{m}}_{r,E,G}(A) < \delta$, we have

$$r(\nu_m(A)) \ \le \ \mathfrak{N}_r(H_m 1_A) \ \le \ \frac{\epsilon}{2} \ \le \ \epsilon \text{ for } m = 1, 2, \ldots, K$$

and

$$\begin{aligned} r(\nu_m(A)) &\le \mathfrak{N}_r(H_m 1_A) \\ &\le \mathfrak{N}_r((H_m - H_K)1_A) + \mathfrak{N}_r(H_K 1_A) \\ &\le \mathfrak{N}_r(H_m - H_K) + \mathfrak{N}_r(H_K 1_A) \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } m > K \ . \end{aligned}$$

□

Proposition 4.2.8. *Let* $\{H_n\}, \{K_n\}$ *be mean Cauchy sequences from* $S_E(\mathscr{S})$*. Suppose that* $H : T \to E$ *is a measurable function and that each sequence converges in* $\boldsymbol{m}_{E,G}$*-measure to* H*. Let* μ_n, ν_n *be the respective indefinite integrals for* H_n, K_n*. Then the set functions* $\mu, \nu : \mathscr{S} \to G$ *defined by* $\mu(A) = \lim \mu_n(A)$ *and* $\nu(A) = \lim \nu_n(A)$ *are identical on* $\mathscr{S}$*.*

Proof. The set functions μ, ν exist, as defined, by proposition 4.2.5. Fix $A \in \mathscr{S}$. Let $r \in \mathscr{G}$ and $\epsilon > 0$. Put

$$A_n \;=\; \{p_r(H_n - K_n) > \epsilon\} \;\in\; \mathscr{S}, \quad n = 1, 2, \ldots$$

Since

$$\begin{aligned} \widehat{\boldsymbol{m}}_{r,E,G}(A_n) \;&\le\; \widehat{\boldsymbol{m}}_{r,E,G}\Big(\{p_r(H_n - H) > \frac{\epsilon}{2}\} \cup \{p_r(K_n - H) > \frac{\epsilon}{2}\}\Big) \\ &\le\; \widehat{\boldsymbol{m}}_{r,E,G}\{p_r(H_n - H) > \frac{\epsilon}{2}\} \;+\; \widehat{\boldsymbol{m}}_{r,E,G}\{p_r(K_n - H) > \frac{\epsilon}{2}\}\,, \end{aligned}$$

we see $\widehat{\boldsymbol{m}}_{r,E,G}(A_n) \to 0$.

Proposition 4.2.7 implies that there is a $\delta > 0$ for which $\mathfrak{N}_r(H_n 1_A) \le \epsilon$ and $\mathfrak{N}_r(K_n 1_A) \le \epsilon$ for all $n \in \mathbb{N}$, whenever $B \in \mathscr{S}$ and $\widehat{\boldsymbol{m}}_{r,E,G}(B) < \delta$. Thus, for our $A \in \mathscr{S}$, we have

$$\begin{aligned} r(\nu_n(A) - \mu_n(A)) \;&\le\; \mathfrak{N}_r((H_n - K_n)1_A) \\ &\le\; \mathfrak{N}_r((H_n - K_n)1_{A-A_n}) + \mathfrak{N}_r(H_n 1_{A\cap A_n}) + \mathfrak{N}_r(K_n 1_{A\cap A_n}) \\ &\le\; \epsilon\,\widehat{\boldsymbol{m}}_{r,E,G}(A - A_n) + \epsilon + \epsilon, \\ &\qquad \text{once } n \text{ is large enough to make } \widehat{\boldsymbol{m}}_{r,E,G}(A_n) < \delta, \\ &\le\; \epsilon\,(\widehat{\boldsymbol{m}}_{r,E,G}(T) + 2)\,. \end{aligned}$$

Since $\widehat{\boldsymbol{m}}_{r,E,G}(T) < \infty$, we have $r(\nu_n(A) - \mu_n(A)) \xrightarrow{n} 0$. As $r \in \mathscr{G}$ was arbitrary,

$$\nu(A) = \lim_{n\to\infty} \nu_n(A) = \lim_{n\to\infty} \mu_n(A) = \mu(A) \text{ in } G\,.$$

□

By proposition 4.2.8, we may now unequivocally define for $H \in \mathfrak{L}_o$ and $A \in \mathscr{S}$:

$$\int_A H \,\mathrm{d}\boldsymbol{m} \;=\; \lim_{n\to\infty} \int_A H_n \,\mathrm{d}\boldsymbol{m} \text{ in } G\,,$$

where $\{H_n\}$ is any determining sequence for H. The set function $\nu : \mathscr{S} \to G$ defined by

$$\nu(A) \;=\; \int_A H \,\mathrm{d}\boldsymbol{m} \;\text{ for }\; A \in \mathscr{S}$$

is called the *indefinite integral* of H. It is additive, and σ-additive when $\boldsymbol{m}$ is σ-additive, by proposition 4.2.5.

For any $A \in \mathscr{S}$, we see that $H1_A$ is determined by $\{H_n 1_A\}$ if $\{H_n\}$ determines H. Thus, we have

$$\int_A H \,\mathrm{d}\boldsymbol{m} \;=\; \lim_{n\to\infty} \int_A H_n \,\mathrm{d}\boldsymbol{m} \;=\; \lim_{n\to\infty} \int H_n 1_A \,\mathrm{d}\boldsymbol{m} \;=\; \int H1_A \,\mathrm{d}\boldsymbol{m} \,.$$

Theorem 4.2.9. $\mathfrak{L} = \mathfrak{L}_o$. *Given any sequence* $\{H_n\} \subset S_E(\mathscr{S})$ *and measurable* H, *the following are equivalent:*

(1) $H \in \mathfrak{L}$ *and* $H_n \to H$ *in mean.*

(2) $\{H_n\}$ *is a determining sequence for* H.

Proof. (1 ⇒ 2) Let $H \in \mathfrak{L}$, and suppose that $H_n \to H$ in mean. Fix $r \in \mathscr{G}$. We have $\mathfrak{N}_r(H_n - H) \xrightarrow{n} 0$. For each $n \in \mathbb{N}$ and $\epsilon > 0$, let

$$A_{n\epsilon} \;=\; \{\, p_r(H_n - H) > \epsilon \,\} \,.$$

Then for each fixed $\epsilon > 0$,

$$\begin{aligned} \mathfrak{N}_r(H_n - H) \;&\geq\; \sup_{z\in U_r{}^\circ} \int p_r((H_n - H)1_{A_{n\epsilon}}) \,\mathrm{d}|\,\boldsymbol{m}_z\,|_r \\ &>\; \epsilon \sup_{z\in U_r{}^\circ} |\,\boldsymbol{m}_z\,|_r(A_{n\epsilon}) \\ &=\; \epsilon\, \widehat{\boldsymbol{m}}_{r,E,G}(A_{n\epsilon}) \,. \end{aligned}$$

Thus, we have $\widehat{\boldsymbol{m}}_{r,E,G}(A_{n\epsilon}) \to 0$. That is, $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure. In addition, by an application of the triangle inequality to the statement $\mathfrak{N}_r(H_n - H) \to 0$, we see that $\{H_n\}$ is mean Cauchy. Hence, $H \in \mathfrak{L}_o$ and $\{H_n\}$ is a determining sequence.

(2 ⇒ 1) Let $H \in \mathfrak{L}_o$, and suppose $\{H_n\}$ determines H. Fix $r \in \mathscr{G}$ and $z \in U_r{}^\circ$. Then for each fixed $\epsilon > 0$, we have that

$$\begin{aligned} |\,\boldsymbol{m}_z\,|_r \{\, |\,p_r(H_n) - p_r(H)\,| > \epsilon \,\} \;&\leq\; |\,\boldsymbol{m}_z\,|_r \{\, p_r(H_n - H) > \epsilon \,\} \\ &\leq\; \widehat{\boldsymbol{m}}_{r,E,G} \{\, p_r(H_n - H) > \epsilon \,\} \xrightarrow{n} 0 \,. \end{aligned}$$

Classically, this means that $p_r(H_n) \to p_r(H)$ in $|\,\boldsymbol{m}_z\,|_r$-measure. Moreover,

$$\begin{aligned}\int |\,p_r(H_n) - p_r(H_m)\,|\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r &\le \int p_r(H_n - H_m)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r \\ &\le \mathfrak{N}_r(H_n - H_m) \overset{m,n}{\to} 0\ .\end{aligned}$$

Since, as previously noted, each $p_r(H_n)$ is classically measurable relative to $|\,\boldsymbol{m}_z\,|_r$, we have that $\{p_r(H_n)\}$ is a determining sequence for $p_r(H)$ relative to $|\,\boldsymbol{m}_z\,|_r$. Hence,

$$\int p_r(H)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r \;=\; \lim_{n\to\infty} \int p_r(H_n)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r \;<\; \infty\ ,$$

for each $z \in U_r{}^\circ$.

Now since $\{H_n\}$ is mean Cauchy, then for each $n \in \mathbb{N}$, we may choose N_n such that $N_1 < N_2 < N_3 < \dots$ and $\mathfrak{N}_r(H_{N_n} - H_m) < \frac{1}{n}$ for $m \ge N_n$. For each $z \in U_r{}^\circ$,

$$\begin{aligned}\int p_r(H_{N_n} - H)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r &= \lim_{m\to\infty} \int p_r(H_{N_n} - H_m)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r \\ &\le \sup_{m\ge N_n} \int p_r(H_{N_n} - H_m)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r \\ &\le \sup_{m\ge N_n} \mathfrak{N}_r(H_{N_n} - H_m) \\ &\le \frac{1}{n}\ .\end{aligned}$$

Taking the supremum over all $z \in U_r{}^\circ$, we see that $\mathfrak{N}_r(H_{N_n} - H) \le \frac{1}{n}$, and as $n \in \mathbb{N}$ was arbitrary, $\mathfrak{N}_r(H_{N_n} - H) \overset{n}{\to} 0$. Additionally, for each $z \in U_r{}^\circ$,

$$\begin{aligned}\int p_r(H)\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r &\le \int p_r(H - H_{N_1})\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r + \int p_r(H_{N_1})\ \mathrm{d}|\,\boldsymbol{m}_z\,|_r \\ &\le \mathfrak{N}_r(H - H_{N_1}) + \mathfrak{N}_r(H_{N_1}) \\ &\le 1 + \mathfrak{N}_r(H_{N_1})\ .\end{aligned}$$

Hence, $\mathfrak{N}_r(H) < \infty$. Since $\{H_n\}$ is mean Cauchy and $N_n \to \infty$ as $n \to \infty$, an application of triangle inequality shows that $\mathfrak{N}_r(H_n - H) \overset{n}{\to} 0$. As $r \in \mathscr{G}$ was arbitrary, we have shown $H \in \mathfrak{F}$, and $H_n \to H$ in mean. Hence, $H \in \mathfrak{L}$. □

Remark 4.2.10. We claim that the requirement of simple functions in the statement of theorem 4.2.9 can be loosened. In fact, the proof does not use the property that the H_n are simple, but for its necessity in order to label a sequence as determining for some H.

Classically, the sequence of H_n need not be simple in order to draw the conclusion

$$\int p_r(H)\, d|\boldsymbol{m}_z|_r \;=\; \lim_{n\to\infty} \int p_r(H_n)\, d|\boldsymbol{m}_z|_r \;<\; \infty\,,$$

but we need only that $\{p_r(H_n)\}$ be $|\boldsymbol{m}_z|_r$-measurable, mean Cauchy, and converge in $|\boldsymbol{m}_z|_r$-measure to $p_r(H)$. These properties follow simply from the fact that $\{H_n\}$ is mean Cauchy and $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure. At the closure of the proof, once we have already shown that $H \in \mathfrak{F}$ and $H_n \to H$ in mean, then provided that $\{H_n\} \subset \mathfrak{L}$, we have $H \in \mathfrak{L}$ simply from the fact that $\mathfrak{L}$ is closed in $\mathfrak{F}$. The remainder of the proof uses inequalities independent of the requirement of simple functions. Thus we have the following equivalence for $\{H_n\} \in \mathfrak{L}$ and measurable H:

(1)′ $H \in \mathfrak{L}$ and $H_n \to H$.

(2)′ $\{H_n\}$ is mean Cauchy and $H_n \xrightarrow{n} H$ in $\boldsymbol{m}_{E,G}$-measure.

Next, in keeping with the theme of extending results for simple functions to integrable functions, we prove the following theorem.

Theorem 4.2.11. *Let $\{H_n\} \subset \mathfrak{L}$ and $H \in \mathfrak{L}$. We have*

$$r\Big(\int_A H\, \mathrm{d}\boldsymbol{m}\Big) \;\le\; \mathfrak{N}_r(H1_A)\,,$$

for each $A \in \mathscr{S}$ and $r \in \mathscr{G}$. If $H_n \xrightarrow{n} H$ in $\mathfrak{L}$, then for each $A \in \mathscr{S}$,

$$\int_A H_n\, \mathrm{d}\boldsymbol{m} \;\xrightarrow{n}\; \int_A H\, \mathrm{d}\boldsymbol{m} \;\; \text{in } G\,.$$

Proof. If H is an element of $S_E(\mathscr{S})$, the first statement is demonstrated by lemma 4.2.4. For a general $H \in \mathfrak{L}$, we obtain a determining sequence of measurable simple functions $\{H_n\}$ for which $H_n \xrightarrow{n} H$ in $\mathfrak{L}$, which is possible by theorem 4.2.9. By definition of the indefinite integral,

$$\int_A H_n\, \mathrm{d}\boldsymbol{m} \;\xrightarrow{n}\; \int_A H\, \mathrm{d}\boldsymbol{m} \;\; \text{in } G$$

for each $A \in \mathscr{S}$. Thus, given any $r \in \mathscr{G}$, the continuity of this r on G implies that

$$\begin{aligned} r\Big(\int_A H \,\mathrm{d}\boldsymbol{m}\Big) &= r\Big(\lim_{n\to\infty}\int_A H_n \,\mathrm{d}\boldsymbol{m}\Big) \\ &= \lim_{n\to\infty} r\Big(\int_A H_n \,\mathrm{d}\boldsymbol{m}\Big) \\ &\le \lim_{n\to\infty} \mathfrak{N}_r(H_n 1_A) \;=\; \mathfrak{N}_r(H 1_A)\ . \end{aligned}$$

To prove the second statement, note now that if $H_n \xrightarrow{n} H$ in $\mathfrak{L}$, then for any $A \in \mathscr{S}$ and for all $r \in \mathscr{G}$,

$$r\Big(\int_A H_n - H \,\mathrm{d}\boldsymbol{m}\Big) \;\le\; \mathfrak{N}_r(H_n - H) \;\xrightarrow{n} 0\ .$$

Since this is shown for all $r \in \mathscr{G}$, we have the result. □

We may now prove the following convergence theorems.

Theorem 4.2.12. *Let $\{H_n\} \subset \mathfrak{L}$, and let $H : T \to E$ be a measurable function. Suppose*

(1) $H_n \to H$ *in $\boldsymbol{m}_{E,G}$-measure*

(2) $\lim\limits_{\widehat{\boldsymbol{m}}_{r,E,G}(\cdot)\to 0} \mathfrak{N}_r(H_n 1_{(\cdot)}) = 0$, *uniformly in n, for each $r \in \mathscr{G}$.*

Then $H \in \mathfrak{L}$ and $H_n \to H$ in mean. Conversely, if $H_n \to H$ in $\mathfrak{L}$, then properties (1) *and* (2) *hold.*

Proof. ($\Rightarrow$) Fix $r \in \mathscr{G}$. Let $\epsilon > 0$. By (2), there is $\delta > 0$ such that $\widehat{\boldsymbol{m}}_{r,E,G}(A) < \delta$ implies $\mathfrak{N}_r(H_n 1_A) < \epsilon$ for each $n = 1, 2, \ldots$ In addition, (1) implies that $\widehat{\boldsymbol{m}}_{r,E,G}(T - B_{mn}) \xrightarrow{m,n} 0$, where

$$B_{mn} \;:=\; \Big\{ p_r(H_n - H_m) \;\le\; \frac{\epsilon}{\widehat{\boldsymbol{m}}_{r,E,G}(T)} \Big\}\ .$$

Thus, there exists M_ϵ for which $\widehat{\boldsymbol{m}}_{r,E,G}(T - B_{mn}) < \delta$ whenever $m, n \ge M_\epsilon$. Then for $m, n \ge M_\epsilon$,

$$\begin{aligned} \mathfrak{N}_r(H_m - H_n) &\le \mathfrak{N}_r((H_m - H_n)1_{B_{mn}}) + \mathfrak{N}_r(H_n 1_{T-B_{mn}}) + \mathfrak{N}_r(H_m 1_{T-B_{mn}}) \\ &\le \Big[\frac{\epsilon}{\widehat{\boldsymbol{m}}_{r,E,G}(T)}\Big]\, \widehat{\boldsymbol{m}}_{r,E,G}(B_{mn}) + \epsilon + \epsilon \\ &\le 3\epsilon\ . \end{aligned}$$

As $\epsilon > 0$ was arbitrary, we have that $\mathfrak{N}_r(H_n - H_m) \xrightarrow{m,n} 0$. Therefore, as $r \in \mathscr{G}$ was arbitrary, $\{H_n\}$ is mean Cauchy, which shows $H_n \to H$ in $\mathfrak{L}$ by remark 4.2.10.

($\Leftarrow$) By remark 4.2.10, (1) is immediate.

Fix $r \in \mathscr{G}$, and let $\epsilon > 0$. Since $H \in \mathfrak{L}$, there is a $K_o \in S_E(\mathscr{S})$ with finite range $R(K_o)$, for which $\mathfrak{N}_r(K_o - H) < \frac{\epsilon}{4}$. Let $M_o = \max_{e \in R(K_o)} p_r(e)$. Then for any $A \in \mathscr{S}$,

$$\begin{aligned} \mathfrak{N}_r(H1_A) &\leq \mathfrak{N}_r((H - K_o)1_A) + \mathfrak{N}_r(K_o 1_A) \\ &\leq \mathfrak{N}_r(H - K_o) + M_o \widehat{\boldsymbol{m}}_{r,E,G}(A) \ . \end{aligned}$$

Hence,

$$\mathfrak{N}_r(H1_A) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \text{ once } \widehat{\boldsymbol{m}}_{r,E,G}(A) < \frac{\epsilon}{4M_o} \ .$$

Now since

$$\mathfrak{N}_r((H_n - H)1_A) \leq \mathfrak{N}_r(H_n - H) \xrightarrow{n} 0$$

for $A \in \mathscr{S}$, there is $N_\epsilon \in \mathbb{N}$ for which

$$\mathfrak{N}_r(H_n 1_A) \leq \mathfrak{N}_r(H1_A) + \frac{\epsilon}{2} \text{ for all } A \in \mathscr{S} \text{ and } n \geq N_\epsilon \ .$$

Hence, for all $n \geq N_\epsilon$,

$$\mathfrak{N}_r(H_n 1_A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ once } A \in \mathscr{S} \text{ and } \widehat{\boldsymbol{m}}_{r,E,G}(A) < \frac{\epsilon}{4M_o} =: \delta \ .$$

Repeat the beginning of this proof of the reverse direction, replacing the function H by the functions $H_1, H_2, \ldots, H_{N_\epsilon - 1}$, to obtain $M_1, M_2, \ldots, M_{N_\epsilon - 1}$ for which

$$\mathfrak{N}_r(H_n 1_A) < \epsilon \text{ whenever } A \in \mathscr{S} \text{ and } \widehat{\boldsymbol{m}}_{r,E,G}(A) < \frac{\epsilon}{4M_n} =: \delta_n \ .$$

Thus, for any $A \in \mathscr{S}$ such that $\widehat{\boldsymbol{m}}_{r,E,G}(A) < \min\{\delta, \delta_1, \ldots, \delta_{N_\epsilon - 1}\}$, we have $\mathfrak{N}_r(H_n 1_A) < \epsilon$ for every $n \in \mathbb{N}$. □

Theorem 4.2.13. *Let $\{H_n\} \subset \mathfrak{L}$ and $K \in \mathfrak{L}$. Let H be a measurable function. Suppose*

1. *$H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure.*
2. *For each $r \in \mathscr{G}$, we have $p_r(H_n) \leq p_r(K)$ for all $n \in \mathbb{N}$.*

Then $H \in \mathfrak{L}$ and $H_n \to H$ in mean.

Proof. Fix $r \in \mathscr{G}$. For each $n \in \mathbb{N}$, we have $\mathfrak{N}_r(H_n 1_A) \leq \mathfrak{N}_r(K 1_A)$ for all $A \in \mathscr{S}$. By taking $K_n = K$ for every $n \in \mathbb{N}$ in theorem 4.2.12, we see $\mathfrak{N}_r(K 1_{(\cdot)}) \ll \widehat{\boldsymbol{m}}_{r,E,G}(\cdot)$. Hence,

$$\lim_{\widehat{\boldsymbol{m}}_{r,E,G}(\cdot) \to 0} \mathfrak{N}_r(H_n 1_{(\cdot)}) = 0 \text{ uniformly for } n = 1, 2, \ldots$$

By the forward direction of theorem 4.2.12, $H_n \to H$ in mean and $H \in \mathfrak{L}$. □

4.3 The Case of Measures with Values in Locally Convex Space

In this section, we extend the theory of integration given in section 4.2 to include, under certain conditions, the case in which the measure takes its values in a locally convex space F. We will be most interested when F satisfies the *special conditions* of Ustunel, which will mean that F is a nuclear space that is complete, reflexive, bornological, and such that its strong dual F' satisfies the same conditions. With these premises, F and F' have neighborhood bases of 0, $\mathscr{U}$ and $\mathscr{U}'$ respectively, such that for each $V \in \mathscr{U}'$, the space $E'(V)$ is a separable Hilbert space, and its separable dual is identified with the Hilbert space $E[V^\circ]$. The analogous result holds for $U \in \mathscr{U}$. Additionally, the sets $\{U^\circ : U \in \mathscr{U}\}$ and $\{V^\circ : V \in \mathscr{U}'\}$ are bases of closed convex balanced bounded sets in F' and F respectively.

We continue to let E, G be complete locally convex spaces having bases of seminorms $\mathscr{E}, \mathscr{G}$ respectively, and we let $\boldsymbol{m} : \mathscr{S} \to F$ be a σ-additive measure on a σ-algebra $\mathscr{S}$ of subsets of some set T, where it is now permissible for F to be a complete locally convex space. We assume the existence of a continuous bilinear map $\Phi : E \times F \to G$ that induces a continuous linear map $\phi : E \to L(F, G)$.

Theorem 4.3.1. *The measure $\boldsymbol{m} : \mathscr{S} \to F$ is bounded. That is, the set $\boldsymbol{m}(\mathscr{S})$ is bounded in F.*

Proof. Let $\mathscr{Q}$ denote a basis of continuous seminorms for F. Fix $q \in \mathscr{Q}$. The canonical map $\zeta_q : F \to F(U_q)$ is continuous and linear. Thus, $\zeta_q \circ \boldsymbol{m} : \mathscr{S} \to F(U_q)$ is a σ-additive measure taking values in the Banach space $F(U_q)$. It is known that the vector measure $\zeta_q \circ \boldsymbol{m}$ is then bounded ([D–S]). That is, there is a number M_q such that for all $A \in \mathscr{S}$,

$$\begin{aligned} M_q &\geq \| [\boldsymbol{m}(A)]_q \|_{F(U_q)} \\ &= q(\boldsymbol{m}(A)) , \end{aligned}$$

where the equality follows from equation (4.1). That is, $\boldsymbol{m}(\mathscr{S}) \subset M_q U_q$. As $q \in \mathscr{Q}$ was arbitrarily chosen, we have the result. □

Suppose now that F is a nuclear locally convex space, with bases of seminorms $\mathscr{Q}, \mathscr{Q}'$ for F, F' respectively, satisfying the special conditions of Ustunel as stated at the opening of the current section.

By theorem 4.3.1, we may choose a seminorm $q' \in \mathscr{Q}'$ for which the closed convex balanced bounded set $B := U_{q'}^{\circ} \supset \boldsymbol{m}(\mathscr{S})$. Algebraically, then, we are able to consider $\boldsymbol{m} : \mathscr{S} \to F[B]$. Since the topology of $F[B]$ is stronger than the topology of F, it is not at all clear whether $\boldsymbol{m}$ remains σ-additive. Theorem 4.3.2 shows that it is, indeed, σ-additive, and uses the requirement that F, F' satisfy the special conditions of Ustunel to presume that $F'(B^{\circ})$ is a Hilbert space and that F is reflexive.

Theorem 4.3.2. *Suppose F is a nuclear locally convex space satisfying the conditions of Ustunel. Let $\boldsymbol{m} : \mathscr{S} \to F$ be a σ-additive measure, and let B be chosen as above. Then $\boldsymbol{m} : \mathscr{S} \to F[B]$ is σ-additive.*

Proof. By a theorem of Pettis [P], it suffices to demonstrate that $\boldsymbol{m} : \mathscr{S} \to F[B]$ is weakly σ-additive. Let $A = \cup_{n=1}^{\infty} A_i$, where $A_i \in \mathscr{S}$ are pairwise disjoint. Note that for any $x' \in F'$, we have

$$x'\boldsymbol{m}\Big(\bigcup_{i=n}^{\infty} A_i \Big) = x'\Big(\boldsymbol{m}(A) - \sum_{i=1}^{n} \boldsymbol{m}(A_i) \Big) \xrightarrow{n} 0 , \tag{4.7}$$

regardless of the arrangement of the A_i, since $\boldsymbol{m} : \mathscr{S} \to F$ is σ-additive. We need to show the same holds for x' in $F[B]'$.

Let $f' \in F[B]'$. Since $F'(B^{\circ})$ is a Hilbert space and since F is reflexive,

$$F[B]' = ((F'(B^{\circ}))')' = F'(B^{\circ}) .$$

Thus, there is an $f'_o \in F'(B^\circ)$ corresponding to f'. Recall that the elements of the form $[x'_o]_{B^\circ}$, where $x'_o \in F'$ are dense in the Hilbert space $F'(B^\circ)$. If f'_o is one of these elements, then the fact that

$$f'(f) \;=\; [x'_o]_B(f) \;=\; x'_o(f)$$

in this correspondence yields (4.7) for f'. In general, we may obtain $[x'_k]_{B^\circ}$, where $x'_k \in F'$, for which $[x'_k]_{B^\circ} \to f'_o$ in $F'(B^\circ)$. Let each $[x'_k]_{B^\circ}$ correspond to f'_k in $F[B]'$. We have that $f'_k - f' \to 0$ in $F[B]'$. That is,

$$\sup_{f \in B} |(f'_k - f')f| \to 0 \, .$$

Thus, given $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that, for all $k \geq K$ and all $f \in B$, we have $|(f'_k - f')f| \leq \epsilon$. Now since (4.7) holds for each f'_k, we may choose $N \in \mathbb{N}$ for which

$$\left| f'_K \boldsymbol{m}\Big(\bigcup_{i=n}^{\infty} A_i \Big) \right| \;\leq\; \epsilon$$

whenever $n \geq N$. Note that every $f_n := \boldsymbol{m}(\cup_{i=1}^{n} A_i)$ is an element of B. Thus, whenever $n \geq N$, we have

$$\begin{aligned} |f'(f_n)| \;&\leq\; |(f'_K - f')(f_n)| \;+\; |f'_K(f_n)| \\ &\leq\; 2\epsilon \end{aligned}$$

for all $n \geq \mathbb{N}$. Hence, (4.7) holds for f'. □

We are almost able to define an integral and Lebesgue space for this case. We would like to utilize our Banach-valued σ-additive measure $\boldsymbol{m} : \mathscr{S} \to F[B]$, reducing the current problem to the known case from the previous section. It remains to find a continuous linear map from the space E, where our functions will take their values, to the space $L(F[B], G)$. As one might expect, we need only treat ϕ as acting on $F[B]$, in place of F, to accomplish this end.

Proposition 4.3.3. *Under the assumptions of theorem 4.3.2, the map $\phi : E \to L(F, G)$ induces a continuous linear map $\phi_B : E \to L(F[B], G)$, defined for $e \in E$ by $(\phi_B(e))f_B = \phi(e)f_B$ for every $f_B \in F[B]$.*

Proof. The linearity of ϕ_B and of $\phi_B(e)$ for each $e \in E$ follows precisely from the definition, and from the linearity of ϕ and each $\phi(e)$.

The map is into, for if a net $f_\alpha \overset{\alpha}{\to} 0$ in $F[B]$, then $f_\alpha \overset{\alpha}{\to} 0$ in F, as the topology of $F[B]$ is stronger than that of F. Since each $\phi(e)$ is continuous for $e \in E$, we have $(\phi_B(e))f_\alpha = \phi(e)f_\alpha \overset{\alpha}{\to} 0$ in G. Hence, $\phi_B(e)$ is continuous for each $e \in E$.

In order to show that ϕ_B is continuous, we recall that as r varies over $\mathscr{G}$, the neighborhoods

$$\mathfrak{U}_r \quad = \quad \{ \zeta \in L(F[B], G) : \zeta(B) \subset U_r \}$$

form a basis of neighborhoods for $L(F[B], G)$, since B is the unit ball of $F[B]$.

Now given $r \in \mathscr{G}$, by the continuity of Φ there exist neighborhoods U_p and U_q of E and F for which $\Phi(U_p, U_q) \subset U_r$. Since B is bounded in F, there is a $\lambda > 0$ such that $\lambda B \subset U_q$. Hence, $\Phi(U_p, \lambda B) \subset U_r$. That is,

$$\phi(\lambda U_p) \quad \subset \quad \{ \zeta : \zeta(B) \subset U_r \} \; .$$

Hence, $\phi^{-1}(\mathfrak{U}_r)$ envelops a neighborhood of 0 in E. As $r \in \mathscr{G}$ was arbitrary, ϕ_B is continuous. ☐

We now have the tools necessary to define an integral and Lebesgue space. Utilizing the ideas presented above, we may define measurability and the spaces of E-valued functions $\mathfrak{F}(\overline{m}_{E,G})$ and $\mathfrak{L}(\overline{m}_{E,G})$ relative to the continuous linear map $\phi_B : E \to L(F[B], G)$ and the σ-additive measure $m_B = \mathscr{S} \to F[B]$, provided the semivariation $(m_B)_{\phi_B, E, G}$ is finite. As the action of $\phi_B(e)$ on $m(\mathscr{S})$ coincides with the action of $\phi(e)$ on $m(\mathscr{S})$, this construction seems only natural.

4.4 Bounded Measurable Functions

In this section, we examine the particular case of bounded measurable functions $H : T \to E$. We first define the integral of such functions in a different, less complicated manner. To do so, we shall require that E be a nuclear locally convex space with basis $\mathscr{E}$ satisfying the special conditions of Ustunel. We then show that, provided G has a countable basis of seminorms and there is no $r \in \mathscr{G}$ for which $G(U_r) \supset c_o$, the bounded measurable functions are elements of the space $\mathfrak{L}(m_{E,G})$ of integrable functions. Finally, we show that the integrals agree in the latter case, and thus the integration outlined in section 4.2 is an extension of this alternate method to unbounded functions.

As demonstrated in section 4.3, if F is nuclear with basis $\mathscr{Q}$ satisfying the conditions of Ustunel, and if $\boldsymbol{m} : \mathscr{S} \to F$ is σ-additive, then we may instead consider the (bounded) σ-additive measure $\boldsymbol{m}_B : \mathscr{S} \to F[B]$, where B is a closed convex balanced bounded set for which $\boldsymbol{m}(\mathscr{S}) \subset B$, to create a Lebesgue space and integral in the manner of section 4.2. The results of this section will then apply to this case. However, for the remainder of the section, we simply assume that F is a Banach space, and for the moment we assume that $\boldsymbol{m} : \mathscr{S} \to F$ is an additive measure.

Given any bounded measurable function $H : T \to E$, we have that the range of H is contained in some closed convex balanced bounded set $B_1 \subset E$. If we take E to be a nuclear locally convex space with basis of seminorms $\mathscr{E}$ satisfying the special conditions of Ustunel, then we may choose B_1 so that $E[B_1]$ is a separable Hilbert space. Since the underlying map $\Phi : E \times F \to G$ is continuous, the set $\Phi(B_1, F_1)$ is bounded in G. Thus, there is a closed convex balanced bounded set $B \subset G$ for which $\Phi(B_1, F_1) \subset B$.

Algebraically, $E[B_1], F, G[B]$ are the spans of the unit balls B_1, F_1, B, respectively, and thus the bilinear nature of Φ demands that $G[B] \supset \Phi(E[B_1], F)$. We may then define, by restricting Φ to $E[B_1] \times F$, a bilinear map $\Phi_o : E[B_1] \times F \to G[B]$. Moreover, Φ_o is continuous since $\Phi_o^{-1}(\alpha B) \supset (\alpha B_1) \times F_1$ for any scalar α. In the usual manner, this map induces a continuous linear map $\phi_o : E[B_1] \to L(F, G[B])$.

Note that for any $e \in E[B_1]$,

$$\begin{aligned} \| \phi_o\Big(\frac{e}{\| e \|_{E[B_1]}} \Big) \|_{L(F,G[B])} &= \sup_{f \in F_1} \| \phi_o\Big(\frac{e}{\| e \|_{E[B_1]}} \Big) f \|_{G[B]} \\ &= \sup_{f \in F_1} \| \Phi_o\Big(\frac{e}{\| e \|_{E[B_1]}}, f \Big) \|_{G[B]} \\ &\leq 1, \quad \text{since } \frac{e}{\| e \|_{E[B_1]}} \in B_1 , \end{aligned}$$

so that

$$\| \phi_o(e) \|_{L(F,G[B])} \leq \| e \|_{E[B_1]} .$$

As two distinct elements of B_1 could conceivably have the same action on the span of F, there is no reason to believe that ϕ_o is an isometry. However, because we extended the

Banach integration theory to continuous linear maps in section 3.1, one could now form the Lebesgue spaces and define integrals for measurable functions $H : T \to E[B]$ relative to ϕ_o, provided that the ϕ_o-semivariation is finite, without ever having seen those constructed in section 4.2. It is a drawback of this method that the finiteness of the ϕ_o-semivariation remains to be shown. Indeed, it is important to recall that the spaces $E[B_1]$ and $G[B]$ are dependent on the function H, and, thus, in order to consider all bounded functions, we would need to consider the finiteness of all manner of such ϕ_o-semivariations. However, this procedure is not always so troublesome. For example, in the case of the stochastic integral in section 6.2, we shall be instead interested in the induced maps $\overline{\phi}_o : E[B_1] \to L(L^2_F, L^2_{G[B]})$, and all such $\overline{\phi}_o$-semivariations will indeed be finite.

We suppose then that the ϕ_o-semivariation is finite. We claim that the bounded measurable function H under consideration will be integrable, that is, an element of $\mathfrak{L}(m_{E[B_1],G[B]})$. In view of theorem 3.1.10, it is enough to know that $H : T \to E[B]$ is bounded and measurable, and that $m_{E[B_1],G[B]}$ is relatively weakly compact in $ca(\mathscr{S})$. In order to achieve the property of weak compactness, we add the assumptions that $G[B] \not\supset c_o$ and that m is σ-additive in accordance with theorem 2.2.5. This is satisfied, for example, if G is a nuclear space satisfying the conditions of Ustunel, and if B is chosen from the basis of bounded sets for G in such a way that $G[B]$ is Hilbert. The measurability and boundedness conditions follow from lemma 4.4.1 below.

Lemma 4.4.1. *Let E be a nuclear space satisfying the conditions of Ustunel, and let $H : S \to E$ a bounded measurable function. Let B_o be any closed convex balanced bounded subset of E that contains the range of H and for which $E[B_o]$ is a separable Hilbert space. Then $H : S \to E[B_o]$ is bounded and measurable.*

Proof. The fact that H is still bounded follows from the fact that its range is contained in B_o, which is the unit ball of $E[B_o]$. Assume that H is the pointwise limit in E of $H_n \in S_E(\mathscr{S})$. Then, for any $x' \in E'$, we have that $x'H_n \to x'H$ pointwise. Because the functions are scalar-valued, we know that for any disk $D_\epsilon \subset \mathbb{C}$ of radius ϵ,

$$(x'H)^{-1}(D_\epsilon) \quad \in \quad \mathscr{S}$$

since each simple function $x'H_n$ has this property. Given $e' \in E[B_o]'$, the reflexivity of E allows us to say

$$E[B_o]' \; = \; ((E'(B_o^\circ))')' \; = \; E'(B_o^\circ)\,.$$

Hence, there is $e'_o \in E'(B_o^\circ)$ related to e' in the above correspondence. Recall that the elements of the form $[x']_{B_o^\circ}$ where $x' \in E'$ are dense in $E'(B_o^\circ)$. If $e'_o = [x'_o]_{B_o^\circ}$ is one of these elements, then for any $e \in E[B_o]$,

$$e'(e) \; = \; [x'_o]_{B_o^\circ}(e) \; = \; x'_o(e)$$

in the above correspondence. Since the range of H is algebraically in $E[B_o]$, we have $e'H = x'_oH$, and hence

$$(e'H)^{-1}(D_\epsilon) \in \mathscr{S} \text{ for all } \epsilon > 0\,.$$

Suppose now, more generally, that $\{[x'_k]_{B_o^\circ}\}$, where $x'_k \in E'$, is a sequence of elements from the dense set for which $[x'_k]_{B_o^\circ} \to e'_o$. Let $e'_k \in E[B_o]'$ correspond to $[x'_k]_{B_o^\circ}$ for each $k \in \mathbb{N}$. Then $e'_k \to e'$ in $E[B_o]'$. That is,

$$\sup_{e \in B_o} |\, e'_k(e) - e'(e)\, | \xrightarrow{k} 0\,.$$

Let $t \in T$. Then $H(t) \in B_o$, and hence $e'_k(H(t)) \xrightarrow{k} e'(H(t))$. That is, as $t \in T$ was chosen arbitrarily, $e'_kH \xrightarrow{k} e'H$ pointwise. Again, the functions involved are scalar-valued, so that

$$(e'H)^{-1}(D_\epsilon) \in \mathscr{S} \text{ for all } \epsilon > 0\,,$$

since the functions e'_k satisfy the same property for each $k \in \mathbb{N}$. Because $e' \in E[B_o]'$ was arbitrarily chosen, we have that $H : S \to E[B_o]$ is weakly measurable, and hence measurable by a theorem due to Pettis [P] since the space $E[B_o]$ is separable. $\square$

We summarize the method thus far in the next remark. We shall refer to it again at a later time in section 6.2.

Remark 4.4.2. Let E be a nuclear locally convex space with basis $\mathscr{E}$ of seminorms satisfying the conditions of Ustunel. Let F be a Banach space, and let $m : \mathscr{S} \to F$ be a σ-additive measure. Let G be a complete locally convex space. Let $H : T \to E$ be a bounded measurable function. For any underlying continuous bilinear map $\Phi : E \times F \to G$, there are closed convex balanced bounded sets $B_1 \subset E$ and $B \subset G$ for which $E[B_1]$ is a separable Hilbert space containing the range of H, and such that the restriction of Φ, written $\Phi_o : E[B_1] \times F \to G[B]$, is a continuous bilinear map that induces a continuous linear map $\phi_o : E[B_1] \to L(F, G[B])$. We assume further that B can be chosen so that $G[B] \not\supset c_o$, and that the ϕ_o-semivariation is finite. Then $H : T \to E[B_1]$ is integrable relative to ϕ_o, that is, $H \in \mathfrak{L}(\boldsymbol{m}_{E[B_1],G[B]})$. Thus, the integral $\int_A H \, d\boldsymbol{m}$ is defined as an element of $G[B]$, and hence algebraically as an element of G, for each $A \in \mathscr{S}$.

We now turn our attention to the method of integration outlined in section 4.2. We assume the hypotheses of that section, and we form the spaces $\mathfrak{F}$, $\mathfrak{L}$, and the indefinite integrals of integrable functions accordingly. We will need three extra assumptions on spaces E and G. We suppose that E is a nuclear locally convex space with basis of seminorms $\mathscr{E}$ satisfying the conditions of Ustunel. We further suppose that the basis of seminorms $\mathscr{G}$ for G is countable, and such that for each $r \in \mathscr{G}$, the space $G(U_r)$ contains no copy of c_o. Finally, we suppose $\boldsymbol{m}$ is σ-additive.

Lemma 4.4.3. *There exists a bounded control measure λ for the set*

$$M_{E,G} = \{ \widehat{\boldsymbol{m}}_{r,E,G} : r \in \mathscr{G} \} .$$

Proof. For each $r \in \mathscr{G}$, the space $G(U_r)$ contains no copy of c_o. Thus, by theorem 2.2.5, each set $\boldsymbol{m}_{r,E,G} = \{ | \boldsymbol{m}_z |_r : z \in U_r{}^\circ \}$ is relatively weakly compact in $ca(\mathscr{S})$. Hence, there are bounded control measures λ_r for which $\boldsymbol{m}_r \overset{\text{unif}}{\ll} \lambda_r \leq \widehat{\boldsymbol{m}}_r$. Since $\mathscr{G}$ is countable, say $\mathscr{G} = \{r_1, r_2, \ldots\}$, then in the usual way, one may find a bounded control measure λ for which $\{\lambda_{r_n}\} \overset{\text{unif}}{\ll} \lambda$ by taking

$$\lambda(A) \; = \; \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_{r_n}(A)}{1 + \lambda_{r_n}(T)} \quad \text{for } A \in \mathscr{S} .$$

Thus, $\{\widehat{\boldsymbol{m}}_r\}_{r \in \mathscr{G}} \overset{\text{unif}}{\ll} \lambda$. □

Now, subject to the conditions placed on E, F, G in the discussion prior to lemma 4.4.3, we have the result that the bounded measurable functions $H : T \to E$ are integrable.

Theorem 4.4.4. *Let $H : \mathbb{R}^+ \times \Omega \to E$ be bounded and measurable. Then $H \in \mathfrak{L}(\boldsymbol{m}_{E,G})$.*

Proof. By lemma 4.4.1, if we let B_o be any closed convex balanced bounded set containing the range of H and such that $E[B_o]$ is Hilbert, then $H : \mathbb{R}^+ \times \Omega \to E[B_o]$ is still bounded and measurable. Let λ be the control measure from theorem 4.4.3. Then H is integrable with respect to λ. Thus, there exist simple measurable functions $H_n : \mathbb{R}^+ \times \Omega \to E[B_o]$ for which $\int \| H_n - H_m \| \, d\lambda \overset{m,n}{\to} 0$, $\{H_n\}$ is Cauchy in λ-measure, and $H_n \to H$ in λ-measure. In addition, the H_n may be chosen so that $\| H_n(t) \| \leq \| H(t) \|$ for each $t \in T$, where $\| \cdot \|$ denotes the $E[B_o]$ norm ([D–S],[D1]).

Fix $r \in \mathscr{G}$ and $\epsilon > 0$. Since the topology of $E[B_o]$ is stronger than the topology induced by E, there exists $\delta_r > 0$ for which $\| e \| < \delta_r$ implies $p_r(e) \leq \epsilon$ for $e \in E[B_o]$. That is, p_r restricted to $E[B_o]$ is continuous, and hence dominated by the norm $\| \cdot \|$. Thus, for all $t \in T$,

$$\| H_n(t) - H(t) \| \leq \delta_r \ \text{ implies } \ p_r(H_n(t) - H(t)) \leq \epsilon \ , \tag{4.8}$$

regardless of $n \in \mathbb{N}$, and hence, for each $n \in \mathbb{N}$,

$$\{ p_r(H_n - H) > \epsilon \} \quad \subset \quad \{ \| H_n - H \| > \delta_r \} \ .$$

Since $\widehat{\boldsymbol{m}}_r \ll \lambda$ and $\lambda \{ \| H_n - H \| > \delta_r \} \to 0$, we have

$$\widehat{\boldsymbol{m}}_r \{ p_r(H_n - H) > \epsilon \} \to 0 \ .$$

That is, $H_n \to H$ in $\boldsymbol{m}_{E,G}$-measure, as $r \in \mathscr{G}$ and ϵ were arbitrarily chosen.

It remains to show $\{H_n\}$ is mean Cauchy, for then $\{H_n\}$ is a determining sequence for H, and we are done.

Let $r \in \mathscr{G}$ be fixed. Regardless of $m, n \in \mathbb{N}$, there is an $M > 0$ such that

$$\| H_n(t) - H(t) \| \ \leq \ 2 \| H(t) \| \ \leq \ M$$

for all $t \in T$, since H has $E[B_o]$-bounded range. Because the restriction of p_r to $E[B_o]$ is continuous, there is a constant K_r for which $e \in E[B_o]$ implies $p_r(e) \leq K_r \,\| e \|$. Hence, for all $m, n \in \mathbb{N}$ and $t \in T$,

$$p_r(\, H_n(t) - H_m(t)\,) \;\leq\; MK_r \; . \tag{4.9}$$

Define $M_r := MK_r$.

Now let $\epsilon > 0$, and choose δ_r as in the first part of the proof. Put

$$A_{mn} \;=\; \{\, \| H_n - H_m \| > \delta_r \,\} \; .$$

Then regardless of $m, n \in \mathbb{N}$, (4.8) and (4.9) yield that

$$\begin{aligned} \mathfrak{N}_r(H_n - H_m) \;&=\; \sup_{z \in U_r{}^\circ} \int p_r(H_n - H_m) 1_{A_{mn}} \,\mathrm{d}|\,\boldsymbol{m}_z\,| \\ &\quad\; + \sup_{z \in U_r{}^\circ} \int p_r(H_n - H_m) 1_{T - A_{mn}} \,\mathrm{d}|\,\boldsymbol{m}_z\,| \\ &\leq\; M_r\, \widehat{\boldsymbol{m}}_r(A_{mn}) + \epsilon\, \widehat{\boldsymbol{m}}_r(\,T - A_{mn}) \\ &\leq\; M_r\, \widehat{\boldsymbol{m}}_r(A_{mn}) + \epsilon\, \widehat{\boldsymbol{m}}_r(\,T) \\ &\leq\; M_r\, \epsilon + \epsilon\, \widehat{\boldsymbol{m}}_r(\,T) \; , \end{aligned}$$

by choosing m, n large enough, since $\widehat{\boldsymbol{m}}_r \ll \lambda$ and $\{H_n\}$ is Cauchy in λ-measure. Since $M_r + \widehat{\boldsymbol{m}}_r(\,T)$ depends only on r, we have $\mathfrak{N}_r(H_n - H_m) \overset{m,n}{\to} 0$. As $r \in \mathscr{G}$ was arbitrarily chosen, $\{H_n\}$ is mean Cauchy. □

Remark 4.4.5. We note something of interest in the proof of theorem 4.4.4. Suppose we attempt to apply the alternate approach to creating the integral, as outlined at the beginning of this section, but with the current added assumptions on E, G, and $\boldsymbol{m}$. Choose any B_o that contains the range of H and for which $E[B_o]$ is a separable Hilbert space; choose any B for which $\Phi(B_o, F_1) \subset B$ and $G[B] \not\supset c_o$. Suppose $\widehat{\boldsymbol{m}} := \widehat{\boldsymbol{m}}_{E[B_o],G[B]}$ is finite, and thus, by theorem 2.2.5, there is a bounded control measure for $\boldsymbol{m}_o := \boldsymbol{m}_{E[B_o],G[B]}$. In the proof, we could have taken λ to be $\lambda = \mu + \mu'$ where $M_{E,G} \ll \mu$ and $\widehat{\boldsymbol{m}} \ll \mu'$. Then, the sequence $\{H_n\}$ obtained thereafter is simultaneously shown to be a determining sequence for H in $\mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$ and a determining sequence for H in $\mathfrak{L}(\boldsymbol{m}_{E[B_o],G[B]})$. We obtain integrals

$\int H \, d\boldsymbol{m}$ in each case. The latter we denote by K_1 and the former by K_2. Since $G[B] \subset G$ by continuous linear embedding with respect to the topologies involved, we have

$$\int H_n \, dX \to K_1 \in G[B] \subset G,$$

and

$$\int H_n \, dX \to K_2 \in G,$$

so that the integrals $K_1 = K_2$ are the same element of G, an element that happens to lie in the span of B. Since this is true regardless of the choice of B_o, B, we see that the integrals for H that may be obtained relative to the various embeddings $\phi_o : E[B_o] \to L(F, G[B])$ by this alternate approach for bounded H are all identical, and in fact, they are identical to the integral obtained by regarding $H \in \mathfrak{L}(\boldsymbol{m}_{E,G})$ in the original, more general approach for all functions in $\mathfrak{L}(\boldsymbol{m}_{E,G})$. The main difference in the alternate approach is that the requirements that G have a countable basis and that $G(U_r) \not\supset c_o$ for each $r \in \mathcal{G}$ are unnecessary. However, one must require the similar condition that B may be chosen so that $G[B] \not\supset c_o$. In each case, one must show or assume finiteness of the respective semivariation(s).

CHAPTER 5
THE SPACE $L^2_G(\boldsymbol{m})$

In this chapter, we develop the space $L^2_G(\boldsymbol{m})$ for a complete locally convex space G, show that it is complete under certain assumptions, and explore its relationship to certain spaces L^2_E, where E is a Banach space. To this end, we first develop the space $L^1_G(\boldsymbol{m})$ in a manner consistent with section 4.2 and examine its properties more closely. The definition of $L^2_G(\boldsymbol{m})$ will then follow naturally. In order to avoid repetition, the entire chapter treats G as defined over the field $\mathbb{C}$ and assumes $\boldsymbol{m}$ to be a $\mathbb{C}$-valued measure. However, one may consistently interchange $\mathbb{C}$ with $\mathbb{R}$ and achieve the same results, which we shall use, in fact, in chapter 6.

5.1 Construction of $L^1_G(\boldsymbol{m})$

Let G be a complete locally convex space. We assume the existence of an additive measure $\boldsymbol{m} : \mathscr{S} \to \mathbb{C}$ for which $|\,\boldsymbol{m}\,| < \infty$, where $\mathscr{S}$ is a σ-algebra of subsets of some set T. Let $\mathscr{G}$ denote a basis of seminorms for G.

We say a function $H : T \to G$ is measurable if H is the pointwise limit in G of simple functions of the form

$$H_n = \sum_{i=1}^{n} g_i 1_{A_i}, \quad \text{where } \; g_i \in G \; \text{ and } \; A_i \in \mathscr{S} \; .$$

Recall that in locally convex space, this means that for each $t \in T$,

$$r(H_n(t) - H(t)) \xrightarrow{n} 0 \;\; \text{for every} \;\; r \in \mathscr{G} \; .$$

As usual, we let $S_G(\mathscr{S})$ denote the space of G-valued simple functions of the above form. For any measurable H and any $r \in \mathscr{G}$, we have that $r(H)$ is $|\,\boldsymbol{m}\,|$-measurable, since it is the pointwise limit of the simple functions $r(H_n)$. We define for each $r \in \mathscr{G}$,

$$N_r(H) \quad = \quad \int r(H) \, \mathrm{d}|\,\boldsymbol{m}\,| \quad \leq \quad \infty \; .$$

Note that one could instead define the seminorms N_r in the style of section 4.2, using the continuous linear injection $\phi : G \to L(\mathbb{C}, G)$ given by $\phi(g)\alpha = \alpha g$ for each $g \in G$ and $\alpha \in \mathbb{C}$. This induces $\phi_r : G(U_r) \to L(\mathbb{C}, G(U_r))$ for each $r \in \mathscr{G}$ in accordance with theorem 4.1.1, since r is certainly the "minimal seminorm p_r for which $U_{p_r} \times D_1 \subset U_r$". Indeed, we have the following calculation from the cited theorem, for each $r \in \mathscr{G}$ and $g \in G$:

$$p_r(g) \;=\; \sup_{|\alpha| \leq 1} r(\phi(g)\alpha) \;=\; \sup_{|\alpha| \leq 1} |\alpha| r(g) \;=\; r(g) \;.$$

But every $\boldsymbol{m}_z$, where $z \in U_r{}^\circ = (G'[U_r{}^\circ])_1$ satisfies

$$|\, \boldsymbol{m}_z |_r = \| z \|_{G'[U_r{}^\circ]} \, |\, \boldsymbol{m} \,| \,, \tag{5.1}$$

so that along with (4.1), we have for any measurable $H : T \to G$,

$$\begin{aligned} \sup_{z \in U_r{}^\circ} \int \| H \|_{G(U_r)} \, \mathrm{d}|\, \boldsymbol{m}_z \,| &= \sup_{z \in U_r{}^\circ} \| z \|_{G'[U_r{}^\circ]} \int r(H) \, \mathrm{d}|\, \boldsymbol{m} \,| \\ &= \int r(H) \, \mathrm{d}|\, \boldsymbol{m} \,| \,, \end{aligned}$$

as noted in remark 2.2.7. Thus, we obtain exactly the N_r defined above. Since, by (5.1), we have

$$\widehat{\boldsymbol{m}}_{r,E,G} \;=\; \sup_{z \in U_r{}^\circ} |\, \boldsymbol{m}_z |_r \;=\; \sup_{z \in U_r{}^\circ} \| z \|_{G'[U_r{}^\circ]} \, |\, \boldsymbol{m} \,| \;=\; |\, \boldsymbol{m} \,| \;<\; \infty \,,$$

we have that the spaces

$$\begin{aligned} \mathfrak{F}^1_G \;&=\; \mathfrak{F}^1_G(\boldsymbol{m}) \;&:=&\; \mathfrak{F}^1_G(\boldsymbol{m}_{E,G}) \\ L^1_G \;&=\; L^1_G(\boldsymbol{m}) \;&:=&\; L^1_G(\boldsymbol{m}_{E,G}) \end{aligned}$$

are already defined, as is the integral of elements in L^1_G, and we obtain the Lebesgue and Vitali convergence theorems as applicable to this situation. The bounded measurable functions are integrable in the event that the following properties hold:

(1) G has a countable basis of seminorms.

(2) $G(U_r) \not\supset c_o$ for each $r \in \mathscr{G}$.

(3) $\boldsymbol{m}$ is σ-additive.

We now summarize the results in this particular setting.

We have

$$\mathfrak{F}^1_G \quad = \quad \{\, \text{measurable } H : N_r(H) < \infty \ \ \text{for each} \ \ r \in \mathscr{G} \,\} \ ,$$

and the space L^1_G is realized as the closure in $\mathfrak{F}^1_G$ of the space of measurable simple functions $S_G(\mathscr{S})$.

We have the following definitions:

Definition 5.1.1. For $H, H_1, H_2, \ldots$ measurable, we say:

(1) $\{H_n\}$ is *mean Cauchy* if $N_r(H_n - H_m) \overset{m,n}{\to} 0$ for each $r \in \mathscr{G}$.

(2) $H_n \to H$ *in mean* if $N_r(H_n - H) \overset{n}{\to} 0$ for each $r \in \mathscr{G}$.

(3) $\{H_n\}$ is *Cauchy in* $|\,\boldsymbol{m}\,|$*-measure* if, for each $r \in \mathscr{G}$ and $\epsilon > 0$, we have $|\,\boldsymbol{m}\,|\{\, r(H_n - H_m) > \epsilon\} \overset{m,n}{\to} 0$.

(4) $H_n \to H$ *in* $|\,\boldsymbol{m}\,|$*-measure* if, for each $r \in \mathscr{G}$ and $\epsilon > 0$, we have $|\,\boldsymbol{m}\,|\{\, r(H_n - H) > \epsilon\} \overset{n}{\to} 0$.

Theorem 5.1.2. *Let $H : T \to G$ be a measurable function. Then $H \in L^1_G$ if and only if H has a determining sequence $\{H_n\} \subset S_G(\mathscr{S})$, defined by*

(1) *$\{H_n\}$ is mean Cauchy.*

(2) *$H_n \to H$ in $|\,\boldsymbol{m}\,|$-measure.*

Moreover, given any sequence $\{H_n\} \subset L^1_G$, the following are equivalent:

(1) *$\{H_n\}$ is mean Cauchy and $H_n \to H$ in $|\,\boldsymbol{m}\,|$-measure.*

(2) *$H \in L^1_G$ and $H_n \to H$ in mean.*

For $H = \sum_{i=1}^n g_i 1_{A_i} \in S_G(\mathscr{S})$ and any $A \in \mathscr{S}$, we have:

$$\int_A H \, d\boldsymbol{m} \quad = \quad \sum_{i=1}^n \boldsymbol{m}(A \cap A_i) g_i \quad = \quad \int H 1_A \, d\boldsymbol{m} \ .$$

More generally, for any $H \in L^1_G$, we have that

$$\int_A H \, d\boldsymbol{m} \quad = \quad \lim_{n\to\infty} \int_A H_n \, d\boldsymbol{m}$$

for any determining sequence $\{H_n\}$, and where the limit is taken in G. The additive set function $\int_{(\cdot)} H \, d\boldsymbol{m}$ is well-defined in this way, and is σ-additive whenever $\boldsymbol{m}$ is σ-additive. Furthermore, we have the following generalization, in accordance with theorem 4.2.11.

Theorem 5.1.3. *Let* $\{H_n\} \subset L^1_G$ *and* $H \in L^1_G$. *We have*

$$r\Big(\int_A H \,\mathrm{d}\boldsymbol{m}\Big) \;\leq\; N_r(H1_A)$$

for each $A \in \mathscr{S}$ *and* $r \in \mathscr{G}$. *If* $H_n \xrightarrow{n} H$ *in* L^1_G, *then for each* $A \in \mathscr{S}$,

$$\int_A H_n \,\mathrm{d}\boldsymbol{m} \;\xrightarrow{n}\; \int_A H \,\mathrm{d}\boldsymbol{m} \quad \text{in } G\,.$$

Finally, we have the following convergence theorems.

Theorem 5.1.4. *Let* $\{H_n\} \subset L^1_G$, *and let* $H : T \to G$ *be a measurable function. Suppose*

(1) $H_n \to H$ *in* $|\boldsymbol{m}|$*-measure.*

(2) $\lim_{|\boldsymbol{m}|(\cdot)\to 0} N_r(H_n 1_{(\cdot)}) = 0$, *uniformly in* n, *for each* $r \in \mathscr{G}$.

Then $H \in L^1_G$ *and* $H_n \to H$ *in mean. Conversely, if* $H_n \to H$ *in* L^1_G, *then properties* (1) *and* (2) *hold.*

Theorem 5.1.5. *Let* $\{H_n\} \subset L^1_G$ *and* $K \in L^1_G$. *Let* H *be a measurable function. Suppose*

(1) $H_n \to H$ *in* $|\boldsymbol{m}|$*-measure.*

(2) *For each* $r \in \mathscr{G}$, *we have* $r(H_n) \leq r(K)$ *for all* $n \in \mathbb{N}$.

Then $H \in L^1_G$ *and* $H_n \to H$ *in mean.*

5.2 Completeness of $L^1_G(\boldsymbol{m})$

The main objective of this section is to show that $L^1_G(\boldsymbol{m})$ is complete when $\boldsymbol{m}$ is σ-additive and G has a countable basis of seminorms $\mathscr{G}$. The results that follow will be used to prove the completeness of L^2_G, as defined in the next section, which will be of great importance to the stochastic integral constructed in section 6.2.

We require two additional definitions:

Definition 5.2.1. Given $H, H_1, H_2, \ldots$ measurable, we say

(1) $H_n \to H$ *almost uniformly* on $B \in \mathscr{S}$ if for each $\epsilon > 0$, there is a set $B_\epsilon \in \mathscr{S}$ with $|\boldsymbol{m}|(B_\epsilon) < \epsilon$, and such that for each $r \in \mathscr{G}$, we have $r(H_n(t) - H(t)) \xrightarrow{n} 0$ uniformly on $B - B_\epsilon$.

(2) $H_n \to H$ *weakly almost uniformly* on $B \in \mathscr{S}$ if for any $r \in \mathscr{G}$ and $\epsilon > 0$, there is a set $B_{r\epsilon} \in \mathscr{S}$ with $|\boldsymbol{m}|(B_{r\epsilon}) < \epsilon$ for which $r(H_n(t) - H(t)) \xrightarrow{n} 0$ uniformly on $B - B_{r\epsilon}$.

The Cauchy versions of (1) and (2) are defined in the usual manner. The main result of this section will now be proved through a series of propositions. We assume that $\boldsymbol{m}$ is σ-additive, although a few of the results hold without that assumption, whereas the requirement that G possesses a countable basis of seminorms is only noted in the statement of those propositions in which it is required.

Proposition 5.2.2. *Let $H, H_1, H_2, \ldots$ be measurable functions, and suppose $H_n \to H$ on $B \in \mathscr{S}$. Then $H_n \to H$ weakly almost uniformly on B.*

Proof. Let $r \in \mathscr{G}$ and $\epsilon > 0$. We define for $k, n \in \mathbb{N}$,

$$B_n^k \;=\; \bigcap_{i=n}^{\infty} \{\, t \in B : r(H_i(t) - H(t)) < \frac{1}{k} \} \;\in\; \mathscr{S} \,.$$

Thus, $B_1^k \subset B_2^k \subset B_3^k \subset \ldots$ and $B \subset \lim\limits_{n\to\infty} B_n^k$ for each $k \in \mathbb{N}$. This implies that $\lim_{n\to\infty} |\boldsymbol{m}|(B - B_n^k) = 0$, so that for each k, we may choose $n_o(k) \in \mathbb{N}$ for which $|\boldsymbol{m}|(B - B_{n_o(k)}^k) < \frac{\epsilon}{2^k}$. Define

$$A \;=\; \bigcup_{k=1}^{\infty} (B - B_{n_o(k)}^k) \;\in\; \mathscr{S}\,;$$

note that $B - A \subset B_{n_o(k)}^k$ for every $k \in \mathbb{N}$. We have

$$|\boldsymbol{m}|(A) \;\le\; \sum_{k=1}^{\infty} |\boldsymbol{m}|(B - B_{n_o(k)}^k) \;<\; \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} \;=\; \epsilon\,.$$

Given $\delta > 0$, choose $k_o \in \mathbb{N}$ so that $\delta > \frac{1}{k_o}$. Thus, once $n > n_o(k_o)$, we have

$$r(H_n(t) - H(t)) \;<\; \frac{1}{k_o} \;<\; \delta \;\text{ for all }\; t \;\in\; B - A \;\subset\; B_{n_o(k_o)}^{k_o}\,.$$

Hence, $r(H_n - H) \xrightarrow{n} 0$ uniformly on $B - A$. □

Proposition 5.2.3. *Let $\{H_n\}$ be a mean Cauchy sequence of measurable functions. Then, $\{H_n\}$ is Cauchy in $|\boldsymbol{m}|$-measure.*

Proof. Given any $r \in \mathscr{G}$ and $\epsilon > 0$, put

$$B_{mnr\epsilon} = \{\, r(H_n - H_m) > \epsilon \,\} \in \mathscr{S} \;\text{ for each }\; m, n \in \mathbb{N}\,.$$

Then for each $m, n \in \mathbb{N}$,

$$\int r(H_n - H_m)\, \mathrm{d}|\boldsymbol{m}| \quad \geq \quad \int_{B_{mnr\epsilon}} r(H_n - H_m)\, \mathrm{d}|\boldsymbol{m}| \quad > \quad \epsilon\, |\boldsymbol{m}|(B_{mnr\epsilon})\ .$$

Since the left side of this equation converges to 0 as $m, n \to \infty$, then we see that $|\boldsymbol{m}|(B_{mnr\epsilon}) \overset{m,n}{\to} 0$ as well. □

Proposition 5.2.4. *Let $\{H_n\}$ be a sequence of measurable functions. Suppose $\{H_n\}$ is almost uniformly Cauchy on T. Then $\{H_n\}$ is Cauchy outside of an $|\boldsymbol{m}|$-negligible set.*

Proof. For each $k \in \mathbb{N}$, there exists $B_k \in \mathscr{S}$ for which $|\boldsymbol{m}|(B_k) < \dfrac{1}{k}$, and such that for every $r \in \mathscr{G}$, we have $r(H_n(t) - H_m(t)) \overset{m,n}{\to} 0$ uniformly on $B - B_k$. Put $B = \cap_{k=1}^{\infty} B_k \in \mathscr{S}$. Then, $|\boldsymbol{m}|(B) \leq |\boldsymbol{m}|(B_k) < \frac{1}{k}$ for all $k \in \mathbb{N}$. That is, $|\boldsymbol{m}|(B) = 0$.

Now given $t \in T - B$, we have $t \in T - B_{k_o}$ for some $k_o \in \mathbb{N}$. Thus, for every $r \in \mathscr{G}$, we have $r(H_n(t) - H_m(t)) \overset{m,n}{\to} 0$. That is, $\{H_n(t)\}$ is Cauchy in G. □

Proposition 5.2.5. *Let $H, H_1, H_2, \ldots$ be measurable functions for which $H_n \to H$ weakly almost uniformly on T. Then $H_n \to H$ in $|\boldsymbol{m}|$-measure.*

Proof. Fix $r \in \mathscr{G}$ and $\epsilon > 0$. Given $\delta > 0$, there is some $B \in \mathscr{S}$ with $|\boldsymbol{m}|(B) < \delta$ for which $r(H_n(t) \to H(t)) \overset{n}{\to} 0$ uniformly outside $T - B$. Thus, we may choose $N \in \mathbb{N}$ for which we have $r(H_n(t) - H(t)) \leq \epsilon$ for all $t \in T - B$, whenever $n \geq N$. Hence,

$$|\boldsymbol{m}|\{\, r(H_n - H) > \epsilon \,\} \quad \leq \quad |\boldsymbol{m}|(B) \quad < \quad \delta \ \text{ whenever } \ n \geq N\ .$$

As $\delta > 0$ was arbitrary, we have $|\boldsymbol{m}|\{\, r(H_n - H) > \epsilon \,\} \overset{n}{\to} 0$. □

Proposition 5.2.6. *Let $\{H_n\}$ be a sequence of measurable functions. Suppose that $\{H_n\}$ is Cauchy in $|\boldsymbol{m}|$-measure. Then there is a subsequence $\{H_{n_k}\}$ that is weakly almost uniformly Cauchy on T.*

Proof. Let $r \in \mathscr{G}$ and $\epsilon > 0$. For each $k \in \mathbb{N}$, obtain $\bar{n}_k \in \mathbb{N}$ for which

$$|\boldsymbol{m}|\{\, r(H_n - H_m) \geq \frac{1}{2^k} \,\} < \frac{1}{2^k} \ \text{ whenever } \ m, n \geq \bar{n}_k\ .$$

Then, choose $n_1 = \bar{n}_1$, $n_2 = \max(1+n_1, \bar{n}_2), \ldots$ so that $\{H_{n_k}\}$ is a subsequence of $\{H_n\}$. For each $s \in \mathbb{N}$, put

$$B_s \;=\; \{\, r(H_{n_s} - H_{n_{s+1}}) \geq \frac{1}{2^s} \,\} \;\in\; \mathscr{S}\,.$$

Note that if $i, j \in \{k, k+1, \ldots\}$ with $i \leq j$, then

$$\begin{aligned} r(H_{n_i}(t) - H_{n_j}(t)) \;&\leq\; \sum_{s=i}^{\infty} r(H_{n_s}(t) - H_{n_{s+1}}(t)) \\ &<\; \sum_{s=i}^{\infty} \frac{1}{2^s} \;=\; \frac{1}{2^{i-1}} \quad \text{outside} \;\; B_k \cup B_{k+1} \cup \ldots \;\in\; \mathscr{S}\,, \end{aligned}$$

and, since $n_s, n_{s+1} \geq \bar{n}_s$ for each $s \in \mathbb{N}$,

$$|\,\boldsymbol{m}\,|(B_k \cup B_{k+1} \cup \ldots) \;\leq\; \sum_{s=k}^{\infty} |\,\boldsymbol{m}\,|(B_s) \;<\; \sum_{s=k}^{\infty} \frac{1}{2^s} \;\leq\; \frac{1}{2^{k-1}}\,.$$

Choose k_o so that $\dfrac{1}{2^{k_o-1}} < \epsilon$. The set $B_{r\epsilon} := B_{k_o} \cup B_{k_o+1} \cup \ldots$ satisfies $|\,\boldsymbol{m}\,|(B_{r\epsilon}) < \epsilon$. Then given any $\delta > 0$, we may choose $i_o \in \{k_o, k_o+1, \ldots\}$ with $\dfrac{1}{2^{i_o-1}} < \delta$, so that for $i, j \geq i_o$,

$$r(H_{n_i} - H_{n_j}) \;<\; \frac{1}{2^{i_o-1}} \;<\; \delta \;\text{ on }\; T - B_{r\epsilon}\,.$$

□

In the event that G has a countable basis of seminorms, we obtain several more useful results, the foundation of which lies in the next proposition.

Proposition 5.2.7. *Suppose G has a countable basis of seminorms, and let $H, H_1, H_2, \ldots$ be measurable functions. If $H_n \to H$ weakly almost uniformly on T, then $H_n \to H$ almost uniformly on T.*

Proof. Let $\mathscr{G}_o = \{r_i\}$ be a countable basis of seminorms. Without loss of generality, we may assume $r_1 \leq r_2 \leq r_3 \leq \ldots$ by forming an equivalent basis from the partial sums of the formal sum $\sum_{i=1}^{\infty} r_i$ of the elements of $\mathscr{G}_o$. Let $\epsilon > 0$. There is, by hypothesis, some set $B_i \in \mathscr{S}$ corresponding to each $r_i \in \mathscr{G}_o$ for which $|\,\boldsymbol{m}\,|(B_i) < \dfrac{\epsilon}{2^i}$, and such that $r_i(H_n - H) \to 0$ uniformly on $T - B_i$. Put $B_\epsilon = \cup_{i=1}^{\infty} B_i \in \mathscr{S}$. Then

$$|\,\boldsymbol{m}\,|(B_\epsilon) \;\leq\; \sum_{i=1}^{\infty} |\,\boldsymbol{m}\,|(B_i) \;<\; \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \;=\; \epsilon\,,$$

and

$$T - B_\epsilon \quad \subset \quad T - B_i \quad \text{for all} \quad i \in \mathbb{N} \ .$$

Hence, $r_i(H_n - H) \to 0$ uniformly on $T - B_\epsilon$ for $i = 1, 2, \ldots$ But, for any $r \in \mathscr{G}$, we have $r \leq K r_{i_o}$ for some $K > 0$ and $r_{i_o} \in \mathscr{G}_o$. Thus, $r(H_n - H) \to 0$ uniformly outside $T - B_\epsilon$. ☐

Proposition 5.2.8. *Suppose G has a countable basis of seminorms, and let $H, H_1, H_2, \ldots$ be measurable functions. If $H_n \to H$ weakly almost uniformly Cauchy on T, then $H_n \to H$ almost uniformly Cauchy on T.*

Proof. The proof is identical to the above proof with the property of convergence replaced by the corresponding Cauchy property. ☐

Proposition 5.2.9. *Suppose G has a countable basis of seminorms and is complete. Let $\{H_n\} \subset S_G(\mathscr{S})$ be Cauchy in $|\boldsymbol{m}|$-measure. Then there exists a measurable function $H : T \to G$ and a subsequence $\{H_{n_k}\}$ for which $H_{n_k} \to H$ in $|\boldsymbol{m}|$-measure.*

Proof. We apply the preceding propositions. By proposition 5.2.6, there is a subsequence $\{H_{n_k}\}$ that it weakly almost uniformly Cauchy. By proposition 5.2.8, $\{H_{n_k}\}$ is almost uniformly Cauchy on T. By proposition 5.2.4, there is a $B \in \mathscr{S}$ for which $|\boldsymbol{m}|(B) = 0$, and such that $\{H_{n_k}\}$ is Cauchy on $T - B$. Since G is complete, $H(t) := \lim\limits_{k \to \infty} H_{n_k}(t)$ exists in G for each $t \in T - B$. For $t \in B$, we put $H(t) = 0$. Thus, $H : T \to G$, and H is measurable, as it is the pointwise limit on T of the sequence $\{H_{n_k} 1_{T-B}\} \subset S_G(\mathscr{S})$.

Now since $H_{n_k} \to H$ on $T - B$, proposition 5.2.2 implies that $H_{n_k} \to H$ weakly almost uniformly on $T - B$. By definition, then, given any $r \in \mathscr{G}$ and $\epsilon > 0$, there is a measurable set $C_{r\epsilon} \in \mathscr{S}$ for which $|\boldsymbol{m}|(C_{r\epsilon}) < \epsilon$, and such that $r(H_n(t) - H(t)) \to 0$ uniformly on $(T - B) - C_{r\epsilon}$. Let $D_{r\epsilon} = B \cup C_{r\epsilon} \in \mathscr{S}$. Then

$$|\boldsymbol{m}|(D_{r\epsilon}) \ \leq \ |\boldsymbol{m}|(B) + |\boldsymbol{m}|(C_{r\epsilon}) \ < \ 0 + \epsilon \ = \ \epsilon \ ,$$

and

$$r(H_n - H) \to 0 \ \text{ uniformly on } \ T - D_{r\epsilon} \ = \ T - (B \cup C_{r\epsilon}) \ = \ (T - B) - C_{r\epsilon} \ .$$

Thus, $H_{n_k} \to H$ weakly almost uniformly on T. By proposition 5.2.5, $H_{n_k} \to H$ in $|\boldsymbol{m}|$-measure. ☐

Theorem 5.2.10. *Suppose the complete locally convex space G has a countable basis of seminorms. Let $\{H_n\} \subset L^1_G$ be a mean Cauchy sequence. Then there exists $H \in L^1_G$ for which $H_n \to H$ in mean. That is, L^1_G is complete.*

Proof. Let $\mathscr{G}_o = \{r_1, r_2, \ldots\}$ be a countable basis of seminorms. Without loss of generality, we may assume $r_1 \leq r_2 \leq r_3 \leq \ldots$ by forming an equivalent basis from the partial sums of the formal sum $\sum_{i=1}^{\infty} r_i$ of the elements of $\mathscr{G}_o$. Note, then, that $N_{r_1} \leq N_{r_2} \leq N_{r_3} \leq \ldots$

Now given any $r \in \mathscr{G}$, there is some number $M > 0$ and some $s_o \in \mathbb{N}$ for which $r \leq Mr_{s_o} \leq Mr_{s_o+1} \leq \ldots$ Fix $n \in \mathbb{N}$. Since $H_n \in L^1_G$, there are simple functions $K_i^{(n)} \in L^1_G$ that converge to H_n in mean. Choose $K_n \in S_G(\mathscr{S})$ for which $N_{r_n}(K_n - H_n) < \dfrac{1}{n}$. Then for any $i, j \in \mathbb{N}$ with $s_o \leq i \leq j$,

$$\begin{aligned} N_r(K_i - K_j) &\leq M\, N_{r_{s_o}}(K_i - K_j) \\ &\leq M\left(N_{r_{s_o}}(K_i - H_i) + N_{r_{s_o}}(H_i - H_j) + N_{r_{s_o}}(H_j - K_j)\right) \\ &\leq M\left(N_{r_i}(K_i - H_i) + N_{r_{s_o}}(H_i - H_j) + N_{r_j}(H_j - K_j)\right) \\ &< M\left(\frac{1}{i} + N_{r_{s_o}}(H_i - H_j) + \frac{1}{j}\right). \end{aligned}$$

Since $N_{r_{s_o}}(H_i - H_j) \overset{i,j}{\to} 0$, we let $i, j \to \infty$ in the above inequality to obtain $N_r(K_i - K_j) \overset{i,j}{\to} 0$. Thus, as $r \in \mathscr{G}$ was arbitrary, the sequence $\{K_n\}$ is mean Cauchy, and hence Cauchy in $|\boldsymbol{m}|$-measure by proposition 5.2.3. By proposition 5.2.9, there is a measurable function $H : T \to G$ and a subsequence $\{K_{n_k}\}$ for which $K_{n_k} \to H$ in $|\boldsymbol{m}|$-measure. But the subsequence $\{K_{n_k}\}$ is still Cauchy in mean, and hence is a determining sequence for H. Thus, by theorem 5.1.2, we have $H \in L^1_G$ and $K_{n_k} \to H$ in mean. Since the entire sequence $\{K_n\}$ is mean Cauchy, an application of triangle inequality shows that $K_n \to H$ in mean.

We must show $H_n \to H$ in mean. But, once again, given a fixed $r \in \mathscr{G}$, there exist $M_o > 0$ and $i_o \in \mathbb{N}$ for which $r \leq M_o r_{i_o}$. Then, whenever $i \geq i_o$, we have

$$\begin{aligned} N_r(H_i - H) &\leq N_r(H_i - K_i) + N_r(K_i - H) \\ &\leq M_o N_{r_{i_o}}(H_i - K_i) + N_r(K_i - H) \\ &\leq M_o N_{r_i}(H_i - K_i) + N_r(K_i - H) \\ &\leq \frac{M_o}{i} + N_r(K_i - H) . \end{aligned}$$

Since we have shown that $N_r(K_i - H) \xrightarrow{i} 0$, we let $i \to \infty$ in the above inequality to obtain $N_r(H_i - H) \xrightarrow{i} 0$. As $r \in \mathscr{G}$ was chosen arbitrarily, $H_i \to H$ in mean. □

5.3 Definition of $L^2_G(\boldsymbol{m})$

Given a measurable function $H : T \to G$, we have that, for every $r \in \mathscr{G}$, the real functions $r(H)$ and $r^2(H)$ are classically $|\boldsymbol{m}|$-measurable, as they are the pointwise limit of measurable simple functions. For every $r \in \mathscr{G}$ and measurable H, we define

$$N_{2,r}(H) = \left[\int r^2(H) \, \mathrm{d}|\boldsymbol{m}| \right]^{\frac{1}{2}} \leq \infty ,$$

and define

$$\mathfrak{F}^2_G = \{ \text{measurable } \ H : N_{2,r}(H) < \infty \ \text{ for every } \ r \in \mathscr{G} \} .$$

We claim that $\mathfrak{F}^2_G$ is a linear manifold, and that $N_{2,r}$ is a seminorm on $\mathfrak{F}^2_G$ for every $r \in \mathscr{G}$. Let $A, B \in \mathfrak{F}^2_G$, and let α be any scalar. We may apply the properties of the space $L^2(|\boldsymbol{m}|)$ to the positive real functions obtained by composition of A and B with r. We have that $r^2(\alpha A) = |\alpha| r^2(A)$ and $[r(A) + r(B)]^2$ are measurable and integrable. Since $r^2(A+B) \leq [r(A) + r(B)]^2$ and $r^2(A+B)$ is measurable, then $r^2(A+B)$ is integrable. Moreover,

$$\begin{aligned} \left[\int r^2(\alpha A) \, \mathrm{d}|\boldsymbol{m}| \right]^{\frac{1}{2}} &= \left[\int |\alpha|^2 r^2(A) \, \mathrm{d}|\boldsymbol{m}| \right]^{\frac{1}{2}} \\ &= |\alpha| \left[\int r^2(A) \, \mathrm{d}|\boldsymbol{m}| \right]^{\frac{1}{2}} , \end{aligned} \tag{5.2}$$

and by Minkowski's inequality,

$$\begin{aligned}\Big[\int r^2(A+B)\ \mathrm{d}|\,\boldsymbol{m}\,|\Big]^{\frac{1}{2}} &\le \Big[\int [\,r(A)+r(B)]^2\ \mathrm{d}|\,\boldsymbol{m}\,|\Big]^{\frac{1}{2}} \\ &\le \Big[\int r^2(A)\ \mathrm{d}|\,\boldsymbol{m}\,|\Big]^{\frac{1}{2}} + \Big[\int r^2(B)\ \mathrm{d}|\,\boldsymbol{m}\,|\Big]^{\frac{1}{2}} \qquad (5.3)\end{aligned}$$

so that αA and $A+B$ are elements of $\mathfrak{F}_G^2$.

Additionally, we claim that for each $r \in \mathscr{G}$, the function $N_{2,r}$ is a seminorm on $\mathfrak{F}_G^2$. Indeed, by (5.3), we have

$$N_{2,r}(A+B) \ \le\ N_{2,r}(A) + N_{2,r}(B)\ ,$$

and furthermore, by (5.2),

$$N_{2,r}(A) \ =\ |\,\alpha\,|\, N_{2,r}(A)\ .$$

Hence, $(\,\mathfrak{F}_G^2, \{N_{2,r}\}\,)$ forms a locally convex space.

Note that if $H \in \mathfrak{F}_G^2$, then for every $r \in \mathscr{G}$, we have that $r^2(H)$ is integrable relative to $|\,\boldsymbol{m}\,|$, that is, classically, $r(H) \in L^2(|\,\boldsymbol{m}\,|)$. Since $|\,\boldsymbol{m}\,| < \infty$, we may apply Hölder inequality to the real functions $r(H)$ and 1_T, to conclude that $r(H) \in L^1(|\,\boldsymbol{m}\,|)$, and

$$\begin{aligned}N_r(H) \ =\ \int r(H)\ \mathrm{d}|\,\boldsymbol{m}\,| &\le \Big[\int r^2(H)\ \mathrm{d}|\,\boldsymbol{m}\,|\Big]^{\frac{1}{2}} \Big[\int 1_T^2\ \mathrm{d}|\,\boldsymbol{m}\,|\Big]^{\frac{1}{2}} \\ &= N_{2,r}(H)\,\sqrt{|\,\boldsymbol{m}\,|(T)}\ . \qquad (5.4)\end{aligned}$$

Hence, $H \in \mathfrak{F}_G^1$, and $\mathfrak{F}_G^2 \subset \mathfrak{F}_G^1$, where the inclusion map is a continuous linear embedding.

Note that the measurable simple functions are in $\mathfrak{F}_G^2$ since $|\,\boldsymbol{m}\,| < \infty$. We define L_G^2 to be the closure of $S_G(\mathscr{S})$ in $\mathfrak{F}_G^2$. Then, again by (5.4), we have that $L_G^2 \subset L_G^1$ by continuous linear embedding. Indeed, if H is measurable and there is a sequence of measurable simple functions $\{H_n\}$ that satisfy $N_{2,r}(H_n - H) \xrightarrow{n} 0$ for each $r \in \mathscr{G}$, then $N_r(H_n - H) \xrightarrow{n} 0$ for each $r \in \mathscr{G}$. Thus, each element of L_G^2 inherits the integral $\int H\ \mathrm{d}\boldsymbol{m}$ from its containment in L_G^1. In particular, if a sequence $\{H_n\} \subset S_G(\mathscr{S})$ is such that $H_n \to H$ in L_G^2, we have

$$\int H\ \mathrm{d}\boldsymbol{m} \ =\ \lim_{n\to\infty} \int H_n\ \mathrm{d}\boldsymbol{m}\ ,$$

where the integral of the simple functions is defined in the usual way, and where the limit is taken in G.

We make the following definitions:

Definition 5.3.1. For $H, H_1, H_2, \ldots$ measurable, we say:

(1) $\{H_n\}$ is *2-mean Cauchy* if $N_{2,r}(H_n - H_m) \overset{m,n}{\to} 0$ for each $r \in \mathscr{G}$.

(2) $H_n \to H$ *in 2-mean* if $N_{2,r}(H_n - H) \overset{n}{\to} 0$ for each $r \in \mathscr{G}$.

Definition 5.3.2. Given $H : T \to E$ measurable, we say H is *2-integrable* if there is a *determining 2-sequence* $\{H_n\} \subset S_E(\mathscr{S})$ defined by

(1) $\{H_n\}$ is 2-mean Cauchy.

(2) $H_n \to H$ in $|\boldsymbol{m}|$-measure.

As is the case for integrable functions in L^1_G, we have that L^2_G is exactly the set of 2-integrable functions, which is demonstrated in the following theorem.

Theorem 5.3.3. *Let $\{H_n\} \subset S_G(\mathscr{S})$ be a sequence of measurable simple functions, and let H be a measurable function. The following are equivalent:*

(1) *$\{H_n\}$ is a determining 2-sequence for H.*

(2) *$H \in L^2_G$ and $H_n \to H$ in 2-mean.*

Proof. ($1 \Rightarrow 2$) Suppose $\{H_n\}$ is a determining 2-sequence for H. Fix $r \in \mathscr{G}$. Since $H_n \to H$ in $|\boldsymbol{m}|$-measure, we have that

$$|\boldsymbol{m}|\{\,|r(H_n) - r(H)\,| > \epsilon\} \;\le\; |\boldsymbol{m}|\{r(H_n - H) > \epsilon\} \overset{n}{\to} 0 \,.$$

Thus, $r(H_n) \to r(H)$ in $|\boldsymbol{m}|$-measure. Since $|\boldsymbol{m}| < \infty$, there exists a subsequence $\{r(H_{n_k})\}$ that converges $|\boldsymbol{m}|$-*a.e.* to H, say on $T - B$, where $B \in \mathscr{S}$ and $|\boldsymbol{m}|(B) = 0$. But then, for any fixed $k \in \mathbb{N}$,

$$|\,r(H_{n_k}(t) - H(t)) - r(H_{n_k}(t) - H_{n_m}(t))\,| \;\le\; r(H_{n_m}(t) - H(t)) \overset{m}{\to} 0\,,$$

for each $t \in T - B$. Thus, for every $t \in T$,

$$r^2(\,H_{n_k}(t) \;-\; H(t)\,)\,1_{T-B}(t) \;=\; \lim_{m \to \infty} r^2(\,H_{n_k}(t) \;-\; H_{n_m}(t)\,)\,1_{T-B}(t)\;.$$

Hence, by Fatou's Lemma,

$$\begin{aligned}\int r^2(H_{n_k} - H)\, \mathrm{d}|\boldsymbol{m}| &= \int r^2(\,(H_{n_k} - H)1_{T-B}\,)\, \mathrm{d}|\boldsymbol{m}| \\ &= \int r^2(H_{n_k} - H)\, 1_{T-B}\, \mathrm{d}|\boldsymbol{m}| \\ &\le \liminf_{m\to\infty} \int r^2(H_{n_k} - H_{n_m})\, 1_{T-B}\, \mathrm{d}|\boldsymbol{m}| \\ &= \liminf_{m\to\infty} \int r^2(H_{n_k} - H_{n_m})\, \mathrm{d}|\boldsymbol{m}|\,. \qquad (5.5)\end{aligned}$$

Since the subsequence $\{H_{n_k}\}$ remains 2-mean Cauchy, the right side of (5.5) is made arbitrarily small by choosing k large. In particular, there exists some $k_o \in \mathbb{N}$ for which $N_{2,r}(H_{n_{k_o}} - H) < \infty$. Hence,

$$N_{2,r}(H) \;\le\; N_{2,r}(H - H_{n_{k_o}}) \;+\; N_{2,r}(H_{n_{k_o}}) \;<\; \infty\,.$$

Moreover, (5.5) shows that $N_{2,r}(H_{n_k} - H) \xrightarrow{k} 0$. As $r \in \mathscr{G}$ was chosen arbitrarily, we have $H \in \mathfrak{F}^2_G$ and $H_{n_k} \to H$ in 2-mean. As the entire sequence $\{H_n\}$ is 2-mean Cauchy, an application of triangle inequality proves that $H_n \to H$ in 2-mean.

(2 $\Rightarrow$ 1) Since $H_n \to H$ in 2-mean, then $H_n \to H$ in mean by (5.4), and $\{H_n\}$ is seen to be 2-mean Cauchy by a standard application of the triangle inequality. But now, since $H_n \to H$ in mean, we have that $H_n \to H$ in $|\boldsymbol{m}|$-measure by theorem 5.1.4. Thus, $\{H_n\}$ is a determining 2-sequence for H. ☐

We now end this section with a result that is vital to the stochastic integral in section 6.2. The proof is similar to the proof of theorem 5.2.10.

Theorem 5.3.4. *Suppose the complete locally convex space G has a countable basis of seminorms. Then L^2_G is complete.*

Proof. Let $\mathscr{G}_o = \{r_1, r_2, \ldots\}$ be a countable basis of seminorms. Without loss of generality, we may assume $r_1 \le r_2 \le r_3 \le \ldots$ by forming an equivalent basis from the partial sums of the formal sum $\sum_{i=1}^{\infty} r_i$ of the elements of $\mathscr{G}_o$. Note, then, that $N_{2,r_1} \le N_{2,r_2} \le N_{2,r_3} \le \ldots$

Now given any $r \in \mathscr{G}$, there exists some number $M > 0$ and some $s_o \in \mathbb{N}$ for which $r^2 \le Mr^2_{s_o} \le Mr^2_{s_o+1} \le \ldots$ Fix $n \in \mathbb{N}$. Since $H_n \in L^2_G$, there are simple functions $K_i^{(n)} \in L_G^2$ that converge to H_n in 2-mean. Choose $K_n \in S_G(\mathscr{S})$ for which $N_{2,r_n}(K_n - H_n) < \dfrac{1}{n}$. Then for any $i, j \in \mathbb{N}$ with $s_o \le i \le j$,

$$\begin{aligned} N_{2,r}(K_i - K_j) &\le M^2 N_{2,r_{s_o}}(K_i - K_j) \\ &\le M^2 \Big(N_{2,r_{s_o}}(K_i - H_i) + N_{2,r_{s_o}}(H_i - H_j) + N_{2,r_{s_o}}(H_j - K_j) \Big) \\ &\le M^2 \Big(N_{2,r_i}(K_i - H_i) + N_{2,r_{s_o}}(H_i - H_j) + N_{2,r_j}(H_j - K_j) \Big) \\ &< M^2 \Big(\frac{1}{i} + N_{2,r_{s_o}}(H_i - H_j) + \frac{1}{j} \Big) . \end{aligned}$$

Since $N_{2,r_{s_o}}(H_i - H_j) \xrightarrow{i,j} 0$, we let $i, j \to \infty$ in the inequality above in order to conclude that $N_{2,r}(K_i - K_j) \xrightarrow{i,j} 0$. Thus, as $r \in \mathscr{G}$ was arbitrary, the sequence $\{K_n\}$ is 2-mean Cauchy, hence mean Cauchy by (5.4). Then by proposition 5.2.3, $\{K_n\}$ is Cauchy in $|\boldsymbol{m}|$-measure. By proposition 5.2.9, there is a measurable function $H : T \to G$ and a subsequence $\{K_{n_k}\}$ for which $K_{n_k} \to H$ in $|\boldsymbol{m}|$-measure. But the subsequence $\{K_{n_k}\}$ is still Cauchy in 2-mean, and hence is a determining 2-sequence for H. Thus, by theorem 5.3.3, we have $H \in L_G^2$ and $K_{n_k} \to H$ in 2-mean. Since the entire sequence $\{K_n\}$ is 2-mean Cauchy, an application of triangle inequality shows that $K_n \to H$ in 2-mean.

We must show $H_n \to H$ in 2-mean. But, once again, given a fixed $r \in \mathscr{G}$, there exist $M_o > 0$ and $i_o \in \mathbb{N}$ for which $r^2 \le M_o r_{i_o}^2$. Then, whenever $i \ge i_o$, we have

$$\begin{aligned} N_{2,r}(H_i - H) &\le N_{2,r}(H_i - K_i) + N_{2,r}(K_i - H) \\ &\le M_o N_{2,r_{i_o}}(H_i - K_i) + N_{2,r}(K_i - H) \\ &\le M_o N_{2,r_i}(H_i - K_i) + N_{2,r}(K_i - H) \\ &\le \frac{M_o}{i} + N_{2,r}(K_i - H) . \end{aligned}$$

Since we have shown that $N_{2,r}(K_i - H) \xrightarrow{i} 0$, we let $i \to \infty$ in the above inequality to obtain $N_{2,r}(H_i - H) \xrightarrow{i} 0$. As $r \in \mathscr{G}$ was chosen arbitrarily, $H_i \to H$ in 2-mean. □

5.4 Banach Space Injections Involving $L_G^2(\boldsymbol{m})$

We continue to let G be a locally convex space with basis of seminorms $\mathscr{G}$.

Given some closed convex balanced bounded set $B \subset G$, or some neighborhood U of 0 in G, we could examine the spaces $L_{G[B]}^2$ or $L_{G(U)}^2$ for the Banach spaces $G[B]$ and $G(U)$.

The remainder of the section is devoted to showing that $L^2_{G[B]} \subset L^2_G$ by continuous linear injection, and $L^2_G(U_{N_{2,r}}) \subset L^2_{G(U_r)}$ for each $r \in \mathscr{G}$ by continuous linear isometry. We begin by demonstrating the latter.

Let $r \in \mathscr{G}$. Given an element $[\,H\,]_{N_{2,r}} \in L^2_G(U_{N_{2,r}})$, where $H \in L^2_G$, we examine the map $[\,H\,]_r : T \to G(U_r)$ defined by $[\,H\,]_r(t) = [H(t)]_r$ for all $t \in T$. We claim that $[\,H\,]_r \in L^2_{G(U_r)}$. For, suppose H_n is a sequence of measurable simple functions for which $H_n \to H$ in L^2_G. Then $N_{2,r}(H_n - H) \to 0$, and the functions $[\,H_n\,]_r$ and $[\,H\,]_r$ are certainly measurable by the continuous linear nature of the quotient map. But, we have

$$\begin{aligned} N^2_{2,r}(H_n - H) &= \int r^2(H_n(t) - H(t))\,\mathrm{d}|\,\boldsymbol{m}\,| \\ &= \int \|\,[\,H_n - H\,]_r(t)\,\|^2_{G(U_r)}\,\mathrm{d}|\,\boldsymbol{m}\,|, \quad \text{by (4.1)}, \\ &= \|\,[\,H_n - H\,]_r\,\|^2_{L^2_{G(U_r)}} \end{aligned} \tag{5.6}$$

Hence, $\{\,[\,H_n\,]_r\}$ is a sequence of measurable simple functions for which $[\,H_n\,]_r \to [\,H\,]_r$ in $L^2_{G(U_r)}$. That is, $[\,H\,]_r \in L^2_{G(U_r)}$.

If $K \in L^2_G$ is another representative for $[\,H\,]_{N_{2,r}}$, then $H - K \in \mathrm{Ker}(N_{2,r})$. That is,

$$0 \;=\; N^2_{2,r}(H - K) \;=\; \|\,[\,H - K\,]_r\,\|_{L^2_{G(U_r)}}\,,$$

in the same manner as (5.6) above. Thus, $[\,H\,]_r = [\,K\,]_r$ in $L^2_{G(U_r)}$.

We may, then, unambiguously define a map $\theta_r \,:\, L^2_G(U_{N_{2,r}}) \to L^2_{G(U_r)}$, first on the elements $[\,H\,]_{N_{2,r}}$ of $L^2_G(U_{N_{2,r}})$, by $\theta_r(\,[\,H\,]_{N_{2,r}}) = [\,H\,]_r$, where $H \in L^2_G$ is any representative.

We have, by replacing $H_n - H$ by H in (5.6) above, and also by (4.1),

$$\|\,[\,H\,]_r\,\|^2_{L^2_{G(U_r)}} \;=\; N^2_{2,r}(H) \;=\; \|\,[\,H\,]_{N_{2,r}}\,\|^2_{L^2_G(U_{N_r})}\,,$$

so that θ_r is a (continuous) isometry. Thus, since $L^2_{G(U_r)}$ is complete, we may extend θ_r as an isometry to the entire space $L^2_G(U_{N_r})$.

Remark 5.4.1. Note that if G is a nuclear space, then $G(U_r)$, and hence $L^2_{G(U_r)}$ may be chosen to be a Hilbert space. Since L^2_G is a locally convex space, but not necessarily nuclear, we know only that $L^2_G(U_{N_r})$ will be a Banach space. However, we have just shown that $L^2_G(U_{N_r})$ is isometric to a subset of $L^2_{G(U_r)}$, and hence it would be, in fact, a Hilbert space.

We now turn our attention to the case of $G[B]$. Recall that, algebraically, $G[B]$ is a subset of G, as it is the span of B in G. The unit ball of $G[B]$ is the closed convex balanced set B, and the norm $\|\cdot\|$ on $G[B]$ is stronger than each seminorm $r \in \mathscr{G}$ restricted to $G[B]$.

Thus, given any $H : T \to G[B]$ in $L^2_{G[B]}$, we have that H is also G-valued; given any $r \in \mathscr{G}$, we have $r \leq K_r \|\cdot\|$ on $G[B]$ for some $K_r > 0$. Therefore, given a sequence $\{H_n\}$ of measurable functions and some measurable function H:

(1) If $\| H_n(t) \to H(t) \| \to 0$, then $r(H_n(t) \to H(t)) \to 0$;

(2) $N^2_{2,r}(H) \;=\; \int r^2(H)\, \mathrm{d}|\boldsymbol{m}| \;\leq\; K^2_r \int \| H \|^2\, \mathrm{d}|\boldsymbol{m}|$;

(3) $N^2_{2,r}(H_n - H) \;\leq\; K^2_r \int \| H_n - H \|^2\, \mathrm{d}|\boldsymbol{m}|$.

Thus, $H : T \to G[B]$ measurable, in $\mathfrak{F}^2_{G[B]}$, or in $L^2_{G[B]}$, imply that $H : T \to G$ is measurable, in $\mathfrak{F}^2_G$, or in L^2_G respectively—that is, when we regard H as a G-valued function. In addition, inequality (2) demonstrates that the unit ball of measurable functions of $L^2_{G[B]}$ is contained in $K^2_r\, U_{N_{2,r}} \cap L^2_{G[B]}$ for each $r \in \mathscr{G}$, and hence the topology of $L^2_{G[B]}$ is stronger than the one induced by L^2_G. Thus, we have $L^2_{G[B]} \subset L^2_G$ by continuous injection (the inclusion map) where $L^2_{G[B]}$ consists of measurable functions.

We summarize these two results, which will play a vital role in the construction of the nuclear stochastic integral in section 6.2.

Theorem 5.4.2. *Let G be a complete locally convex space with basis of seminorms $\mathscr{G}$. Given any $r \in \mathscr{G}$, the map $\theta_r : L^2_G(U_{N_{2,r}}) \to L^2_{G(U_r)}$, defined for all $t \in T$ by*

$$\theta_r(\,[\,H\,]_{N_{2,r}}\,)(t) \;=\; [\,H\,]_r(t) \;:=\; [\,H(t)\,]_r$$

is a continuous linear isometry into.

If B is any closed convex balanced subset of G, then we have $L^2_{G[B]} \subset L^2_G$ (the inclusion map) is a continuous linear injection, where $L^2_{G[B]}$ consists of measurable functions.

CHAPTER 6
THE NUCLEAR STOCHASTIC INTEGRAL

In this chapter, we define the stochastic integral and a Lebesgue space for certain processes taking values in a nuclear locally convex space. Before we can produce this integral, some preliminary results in locally convex space are required.

6.1 The Map $\overline{\phi}$

Let E and G be complete locally convex spaces, and let F be a Banach space. We assume that there is a continuous bilinear map $\Phi : E \times F \to G$, which induces a continuous linear map $\phi : E \to L(F, G)$ given by $\phi(e)f = \Phi(e, f)$. The space $L^2_G(\boldsymbol{m})$ is defined in section 5.3, where G is a complete locally convex space and $\boldsymbol{m} : \mathscr{S} \to F$ is an additive measure of finite variation defined on the σ-algebra $\mathscr{S}$ of subsets of some set T.

Theorem 6.1.1. *The continuous linear map ϕ, defined as above, induces a continuous linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$, defined by $(\overline{\phi}(e)\ell)(t) = \phi(e)(\ell(t))$ for all $\ell \in L^2_F$ and all $t \in T$. Furthermore, $\overline{\phi}$ is injective when ϕ is injective.*

Proof. We first demonstrate that the map is into. Let $\ell \in L^2_F$ with determining 2-sequence $\{\ell_n\}$ of measurable simple functions. Note that for each $e \in E$, the map $\overline{\phi}(e)\ell_n : T \to G$ is a measurable simple function due to the continuous linear nature of $\phi(e)$. Given a seminorm $r \in \mathscr{G}$, we have

$$\begin{aligned}
N_{2,r}(\overline{\phi}(e)\ell) &= \int r^2(\overline{\phi}(e)\ell)(t)\, \mathrm{d}|\boldsymbol{m}| \\
&= \int \| [\phi(e)(\ell(t))]_r \|^2_{G(U_r)}\, \mathrm{d}|\boldsymbol{m}|\,, \quad \text{by (4.1),} \\
&= \int \| \phi_{p_r}([e]_{p_r})\ell(t) \|^2_{G(U_r)}\, \mathrm{d}|\boldsymbol{m}|\,, \quad \text{by theorem 4.1.1,} \\
&\le \int \| \phi_{p_r}([e]_{p_r}) \|^2_{L(F,G(U_r))} \| \ell(t) \|^2_F\, \mathrm{d}|\boldsymbol{m}| \\
&= \int \| [e]_{p_r} \|^2_{E(U_{p_r})} \| \ell(t) \|^2_F\, \mathrm{d}|\boldsymbol{m}|\,, \quad \text{by theorem 4.1.1,} \\
&= \| [e]_{p_r} \|^2_{E(U_{p_r})} \int \| \ell(t) \|^2_F\, \mathrm{d}|\boldsymbol{m}|\,, \qquad (6.1)
\end{aligned}$$

where p_r is the reducing seminorm for r from theorem 4.1.1. Thus, $N_{2,r}(\overline{\phi}(e)\ell) < \infty$, and replacing ℓ by $\ell_n - \ell$ in calculation (6.1) shows that $N_{2,r}(\,\overline{\phi}(e)\ell_n - \overline{\phi}(e)\ell\,) \to 0$. Since $r \in \mathscr{G}$ was arbitrary, $\{\overline{\phi}(e)\ell_n\}$ is a determining 2-sequence for $\overline{\phi}(e)\ell$, so that $\overline{\phi}(e)\ell \in L^2_G$.

We have that $\overline{\phi}(e)$ is linear. For, given $\ell_1, \ell_2 \in L^2_F$ and scalar α,

$$\begin{aligned} \overline{\phi}(e)(\alpha\ell_1 + \ell_2)(t) &= \phi(e)((\alpha\ell_1 + \ell_2)(t)) \\ &= \phi(e)(\alpha\ell_1(t) + \ell_2(t)) \\ &= \alpha\phi(e)(\ell_1(t)) + \phi(e)(\ell_2(t)) \\ &= (\alpha\overline{\phi}(e)\ell_1 + \overline{\phi}(e)\ell_2)(t) \end{aligned}$$

for all $t \in T$, by the linearity of $\phi(e)$ as a map from F to G.

Next, fix $e \in E$. If ℓ_α is a net for which $\ell_\alpha \overset{\alpha}{\to} 0$ in L^2_F, then replacing ℓ by each ℓ_α in calculation (6.1) shows that $N_{2,r}(\overline{\phi}(e)\ell_\alpha) \overset{\alpha}{\to} 0$ for each $r \in \mathscr{G}$. By definition, it follows that $\overline{\phi}(e)\ell_\alpha \overset{\alpha}{\to} 0$ in L^2_G, which proves $\overline{\phi}(e)$ is continuous. Hence, we have shown $\overline{\phi}(e) \in L(L^2_F, L^2_G)$, so that $\overline{\phi}$ is into.

The map $\overline{\phi}$ is seen to be injective, provided ϕ is injective, by examining the maps $\ell_f \in L^2_F$, defined for each $f \in F$ by $\ell_f(t) \equiv f$ for all $t \in T$. Note that every such ℓ_f is in L^2_F since it is simple and measurable. Then if $\overline{\phi}(e) = \overline{\phi}(\bar{e})$ for some $e, \bar{e} \in E$, we have $\overline{\phi}(e)(\ell_f) = \overline{\phi}(\bar{e})(\ell_f)$ for each $f \in F$. Hence, for all $t \in T$, we must have

$$\phi(\bar{e})f \;=\; \phi(\bar{e})(\ell_f(t)) \;=\; \phi(e)(\ell_f(t)) \;=\; \phi(e)f$$

for each $f \in F$. Thus, $\phi(\bar{e}) = \phi(e)$ in $L(F,G)$, so that $\bar{e} = e$ since ϕ is injective.

The map $\overline{\phi}$ is linear. For given $e, \bar{e} \in E$ and scalar α, we have

$$\begin{aligned} (\overline{\phi}(\alpha e + \bar{e})\ell)(t) &= \phi(\alpha e + \bar{e})(\ell(t)) \\ &= [\,\alpha\phi(e) + \phi(\bar{e})\,]\,\ell(t) \\ &= \alpha\phi(e)(\ell(t)) + \phi(\bar{e})(\ell(t)) \\ &= (\alpha\overline{\phi}(e)\ell)(t) + (\overline{\phi}(\bar{e})\ell)(t) \\ &= (\alpha\overline{\phi}(e)\ell + \overline{\phi}(\bar{e})\ell)(t) \end{aligned}$$

for each $\ell \in L^2_F$ and $t \in T$. Thus,

$$\begin{aligned} \overline{\phi}(\alpha e + \bar{e})\ell &= \alpha\overline{\phi}(e)\ell + \overline{\phi}(\bar{e})\ell \\ &= (\alpha\overline{\phi}(e) + \overline{\phi}(\bar{e}))\ell \end{aligned}$$

for each $\ell \in L^2_F$, so that

$$\overline{\phi}(\alpha e + \bar{e}) = \alpha\overline{\phi}(e) + \overline{\phi}(\bar{e}) \ .$$

Finally, $\overline{\phi}$ is continuous. The sets

$$\mathfrak{U}(\,(L^2_F)_1, U_{N_{2,r}}\,) = \{\, \Psi \in L(L^2_F, L^2_G) : \Psi((L^2_F)_1) \subset U_{N_{2,r}} \,\}$$

form a basis of neighborhoods for $L(L^2_F, L^2_G)$ as r varies over $\mathscr{G}$ as defined in Treves [T]. Let $U_{N_{2,r}}$ be a neighborhood for L^2_G, where $r \in \mathscr{G}$. We will show that

$$U_{p_r} \subset (\overline{\phi})^{-1}(\,\mathfrak{U}(\,(L^2_F)_1, U_{N_{2,r}}\,)\,) \ ,$$

which proves continuity. Let $e \in U_{p_r}$. For any $\ell \in L^2_F$ and $t \in T$, we have

$$\begin{aligned} r(\phi(e)(\ell(t))) &= \|\,[\,\phi(e)(\ell(t))\,]_r\,\|_{G(U_r)} \ , \quad \text{by (4.1)}, \\ &= \|\,\phi_{p_r}(\,[\,e\,]_{p_r})\ell(t)\,\|_{G(U_r)} \ , \quad \text{by theorem 4.1.1}, \\ &\leq \|\,\phi_{p_r}(\,[\,e\,]_{p_r})\,\|_{L(F,G(U_r))}\,\|\,\ell(t)\,\|_F \\ &= \|\,[\,e\,]_{p_r}\,\|_{E(U_{p_r})}\,\|\,\ell(t)\,\|_F \ , \quad \text{by theorem 4.1.1}, \\ &= p_r(e)\|\,\ell(t)\,\|_F \ , \quad \text{by (4.1)}\,. \end{aligned} \tag{6.2}$$

Thus, for any $\ell \in (L^2_F)_1$,

$$\begin{aligned} N_{2,r}(\overline{\phi}(e)\ell) &= \int r^2(\phi(e)\ell(t))\ \mathrm{d}|\,\boldsymbol{m}\,| \\ &\leq p^2_r(e)\int \|\,\ell(t)\,\|^2_F\ \mathrm{d}|\,\boldsymbol{m}\,| \\ &\leq 1\ . \end{aligned} \tag{6.3}$$

Thus, $\overline{\phi}(e) \in \mathfrak{U}(\,(L^2_F)_1, U_{N_{2,r}}\,)$. As $e \in U_{p_r}$ was arbitrary, we have the result. □

We proceed to close the discussion of $\overline{\phi}$ with two very interesting and important results.

Remark 6.1.2. Given the map $\phi : E \to L(F,G)$ and a seminorm $r \in \mathscr{G}$, we may utilize ϕ in two distinct directions.

(1) We may first form the map

$$\phi_r : E(U_{p_r}) \ \to \ L(F, G(U_r)) \ ,$$

where p_r is the reducing seminorm for r from theorem 4.1.1, and then form the map

$$\overline{(\phi_r)} : E(U_{p_r}) \ \to \ L(L^2_F, L^2_{G(U_r)}) \ .$$

(2) We may first form the map

$$\overline{\phi} : E \to L(L^2_F, L^2_G) \ ,$$

and then form the map

$$(\overline{\phi})_{N_{2,r}} : E(U_{p_{N_{2,r}}}) \to L(L^2_F, L^2_G(U_{N_{2,r}})) \ ,$$

where $p_{N_{2,r}}$ is the reducing seminorm for $N_{2,r}$ from theorem 4.1.1.

We know p_r and $p_{N_{2,r}}$ are continuous seminorms on E. We claim that

(1) $p_{N_{2,r}} = p_r$.

(2) If $\boldsymbol{m} : \mathscr{S} \to L^2_F$ is an additive measure, then the semivariations

$$\widehat{\boldsymbol{m}}_{1,r} := \widehat{\boldsymbol{m}}_{\overline{(\phi_r)}, E(U_{p_r}), L^2_{G(U_r)}} \quad \text{and} \quad \widehat{\boldsymbol{m}}_{2,r} := \widehat{\boldsymbol{m}}_{(\overline{\phi})_{N_{2,r}}, E(U_{p_{N_{2,r}}}), L^2_G(U_{N_{2,r}})}$$

are identical on $\mathscr{S}$.

Recall that θ_r, which is defined in theorem 5.4.2, provides a continuous linear isometry $\theta_r : L^2_G(U_{N_{2,r}}) \to L^2_{G(U_r)}$, and yields some insight as to why (1) and (2) might hold. We prove these statements in propositions 6.1.3 and 6.1.4 below.

Proposition 6.1.3. *In the notation of remark 6.1.2, the reducing seminorms $p_{N_{2,r}}$ and p_r are identical seminorms on E for each $r \in \mathscr{G}$.*

Proof. Recall from theorem 4.1.1 that

$$p_r \text{ is the infimum of } p \in \overline{\mathscr{E}} \text{ for which } \Phi(U_p, F_1) \subset U_r \ ,$$

and

$$p_{N_{2,r}} \text{ is the infimum of } p \in \overline{\mathscr{E}} \text{ for which } \overline{\Phi}(U_p, (L^2_F)_1) \subset U_{N_{2,r}} \; .$$

If p is such that $\Phi(U_p, F_1) \subset U_r$, then $p_r \leq p$. Given $e \in U_p$ and $\ell \in (L^2_F)_1$, we have, by replacing p_r by p in (6.2) and (6.3) that $N_{2,r}(\overline{\phi}(e)\ell) \leq 1$. Thus, $\overline{\Phi}(U_p, (L^2_F)_1) \subset U_{N_{2,r}}$. Hence, $p_r \geq p_{N_{2,r}}$.

Now suppose that $\overline{\Phi}(U_p, (L^2_F)_1) \subset U_{N_{2,r}}$. That is,

$$p(e) \;\leq\; 1 \quad \text{and} \quad \int \| \ell \|^2_F \, \mathrm{d}|\boldsymbol{m}| \;\leq\; 1$$

imply

$$\int r^2(\phi(e)\ell(t)) \, \mathrm{d}|\boldsymbol{m}| \;=\; N_{2,r}(\overline{\phi}(e)\ell) \;\leq\; 1$$

for $e \in E$ and $\ell \in L^2_F$. For each $f \in F_1$, put

$$\ell_f(t) \;\equiv\; f_T \;:=\; \frac{f}{\sqrt{|\boldsymbol{m}|(T)}}$$

for all $t \in T$. Then

$$\int \| \ell_f(t) \|^2_F \, \mathrm{d}|\boldsymbol{m}| \;\leq\; 1 \, .$$

Thus, if $e \in U_p$, we have

$$\begin{aligned} 1 \;\geq\; \int r^2(\phi(e)\ell_f(t)) \, \mathrm{d}|\boldsymbol{m}| \;&=\; \int r^2(\phi(e) f_T) \, \mathrm{d}|\boldsymbol{m}| \\ &=\; r^2(\phi(e) f_T) \int 1 \, \mathrm{d}|\boldsymbol{m}| \;=\; r^2(\phi(e) f) \, . \end{aligned}$$

Hence, $r(\phi(e)f) \leq 1$. That is $\phi(e)f \in U_r$. Thus, $p_r \leq p_{N_{2,r}}$. ☐

Proposition 6.1.4. *In the notation of remark 6.1.2, we have that* $\widehat{\boldsymbol{m}}_{1,r} = \widehat{\boldsymbol{m}}_{2,r}$ *on* $\mathscr{S}$.

Proof. By definition, for $A \in \mathscr{S}$

$$\widehat{\boldsymbol{m}}_{1,r}(A) \;=\; \sup_{\substack{A_i \amalg A \\ e_i \in E(U_{p_r})_1}} \| \sum_{i=1}^{n} [\, \overline{(\phi_r)}(e_i) \,] \, \boldsymbol{m}(A_i) \, \|_{L^2_{G(U_r)}} \; ,$$

and

$$\begin{aligned}
\widehat{\boldsymbol{m}}_{2,r}(A) &= \sup_{\substack{A_i \amalg A \\ e_i \in E(U_{p_{N_{2,r}}})_1}} \Big\| \sum_{i=1}^{n} [\, (\overline{\phi})_{N_{2,r}}(e_i) \,] \, \boldsymbol{m}(A_i) \Big\|_{L^2_G(U_{N_{2,r}})} \\
&= \sup_{\substack{A_i \amalg A \\ e_i \in E(U_{p_r})_1}} \Big\| \theta_r \Big[\sum_{i=1}^{n} [\, (\overline{\phi})_{N_{2,r}}(e_i) \,] \, \boldsymbol{m}(A_i) \Big] \Big\|_{L^2_{G(U_r)}} \\
&= \sup_{\substack{A_i \amalg A \\ e_i \in E(U_{p_r})_1}} \Big\| \sum_{i=1}^{n} \theta_r \Big[[\, (\overline{\phi})_{N_{2,r}}(e_i) \,] \, \boldsymbol{m}(A_i) \Big] \Big\|_{L^2_{G(U_r)}} ,
\end{aligned}$$

since θ_r is a continuous linear isometry and since $p_r = p_{N_{2,r}}$ by proposition 6.1.3. Thus, it suffices to show that

$$[\, \overline{(\phi_r)}(e) \,] (\ell) \;=\; \theta_r \Big[[\, (\overline{\phi})_{N_{2,r}}(e) \,] (\ell) \Big] \quad \text{in } L^2_{G(U_r)} \tag{6.4}$$

for all $e \in E(U_{p_r})$ and $\ell \in L^2_F$. In fact, in suffices to show (6.4) for elements of the form $[\, e \,]_{p_r}$ where $e \in E$, since such elements are dense in $E(U_{p_r})$, and because the maps $\overline{(\phi_r)}$, $(\overline{\phi})_{N_{2,r}}$, and θ_r are continuous and linear.

Choose any $e \in E$ and $\ell \in L^2_F$. We show the maps in (6.4) agree on each $t \in T$. First, by definition of $(\overline{\phi})_{N_{2,r}}$,

$$\begin{aligned}
\Big[\theta_r \Big[[\, (\overline{\phi})_{N_{2,r}}([\, e \,]_{p_r}) \,] (\ell) \Big] \Big](t) &= \Big[\theta_r \Big([\, (\overline{\phi}(e))\ell \,]_{N_{2,r}} \Big) \Big](t) \\
&= [\, \overline{\phi}(e)\ell \,]_r(t) , \quad \text{by definition of } \theta_r , \\
&= [\, (\overline{\phi}(e)\ell)(t) \,]_r \\
&= [\, \phi(e)\ell(t) \,]_r , \quad \text{by definition of } \overline{\phi} .
\end{aligned}$$

On the other hand, by the definitions of $\overline{(\phi_r)}$ and ϕ_r,

$$\begin{aligned}
\Big[[\, \overline{(\phi_r)}([\, e \,]_{p_r}) \,] (\ell) \Big](t) &= [\, \phi_r([\, e \,]_{p_r}) \,] (\ell(t)) \\
&= [\, \phi(e)\ell(t) \,]_r .
\end{aligned}$$

Thus, (6.4) holds and the result is proved. □

6.2 Construction of the Nuclear Stochastic Integral

In this section, let E and G be nuclear locally convex spaces over the real field $\mathbb{R}$ with bases of seminorms $\mathscr{E}, \mathscr{G}$ satisfying the conditions of Ustunel. These bases will remain fixed throughout the discussion. We have an underlying probability space $(\Omega, \mathscr{F}, \mathcal{P})$ and a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions. We suppose that $\mathscr{G}$ is a countable basis of seminorms, so that $L^2_G(\mathcal{P})$ is complete by theorem 5.3.4. Additionally, we have a square-integrable martingale $X : \mathbb{R}^+ \times \Omega \to F$, where F, F' are nuclear locally convex spaces over the real field $\mathbb{R}$ with bases of seminorms $\mathscr{Q}, \mathscr{Q}'$ satisfying the conditions of Ustunel. It is known that the range of X may be absorbed into some $F[B]$, where $B = U_q{}^{\circ}$ for some $q' \in \mathscr{Q}'$. Henceforth we shall, then, assume that $F = F[B]$ is a Hilbert space. Finally, we assume the existence of a continuous bilinear map $\Phi : E \times F \to G$ that induces a continuous linear map $\phi : E \to L(F, G)$, given by $\phi(e)f = \Phi(e, f)$. This, in turn, induces a continuous, linear map $\overline{\phi} : E \to L(L^2_F, L^2_G)$, as in theorem 6.1.1.

As has been seen in section 3.2, since $F \not\supset c_o$, we may form I_X in the usual manner on the predictable rectangles, and extend to a σ-additive measure, which we also denote $I_X : \mathscr{P} \to L^2_F$, where $\mathscr{P}$ is the σ-algebra generated by the predictable rectangles. It is crucial to note that, by remark 3.2.5, the definition of I_X is independent of any embedding ϕ, and that for every continuous linear embedding $K_1 \to L(F, K_2)$, where K_1 is a Banach space and K_2 is a Hilbert space, the measure I_X has finite semivariation relative to $K_1 \to L(L^2_F, L^2_{K_2})$. Denote $\boldsymbol{m} := I_X$.

Now for each $r \in \mathscr{G}$, the map $\phi_r : E(U_{p_r}) \to L(F, G(U_r))$ will induce the map $\overline{(\phi_r)} : E(U_{p_r}) \to L(L^2_F, L^2_{G(U_r)})$. Since $E(U_{p_r})$ is a Banach space, and $G(U_r)$ is a Hilbert space, we have that the $\overline{(\phi_r)}$-semivariation is finite, due to proposition 3.2.4. But by proposition 6.1.4, we have that the $\overline{(\phi_r)}$-semivariation coincides with the $(\overline{\phi})_r$-semivariation, where $(\overline{\phi})_r : E(U_{p_{N_{2,r}}}) \to L(L^2_F, L^2_G(U_{N_{2,r}}))$ via the reducing seminorm $p_{N_{2,r}}$. Since the seminorms $N_{2,r}$, which were defined in section 5.3, form a basis of seminorms for L^2_G as r varies over $\mathscr{G}$, we see that the $(\overline{\phi})_r$-semivariations are exactly the semivariations $\widehat{\boldsymbol{m}}_{N_{2,r}} = \widehat{\boldsymbol{m}}_{N_{2,r},E,L^2_G} < \infty$ from which, relative to $\overline{\phi}$, we may define the space $\mathfrak{F}(\boldsymbol{m}_{E,L^2_G})$, the space $\mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$, and the indefinite integrals of the integrable processes $H : \mathbb{R}^+ \times \Omega \to E$ as in section 4.2.

We summarize the results in this setting below. Note the importance of having a countable basis of seminorms for G in order to conclude that L^2_G is complete. Otherwise, if the limits of proposition 4.2.5 were taken in the completion of L^2_G, then the resulting integrals may pass from the realm of functions. In addition, note that since each $L^2_G(U_{N_{2,r}})$ is a Hilbert space and therefore contains no copy of c_o, we have that the set

$$\boldsymbol{m}_{N_{2,r}} \;=\; \boldsymbol{m}_{N_{2,r},E,L^2_G(U_{N_{2,r}})} \;=\; \{\, |\,\boldsymbol{m}_z\,|_{N_{2,r}} : z \in U^\circ_{N_{2,r}}\}$$

is relatively weakly compact in $ca(\mathscr{P})$ for each $r \in \mathscr{G}$ by theorem 2.2.5. In fact, since G, and hence L^2_G, has a countable basis of seminorms, then there exists a single control measure λ for which $\widehat{\boldsymbol{m}}_r \ll \lambda$ for every $r \in \mathscr{G}$ by lemma 4.4.3. From this point forward, we shall write $|\,\boldsymbol{m}_z\,|_r$, $\widehat{\boldsymbol{m}}_r$, $\boldsymbol{m}_r$ in place of $|\,\boldsymbol{m}_z\,|_{N_{2,r}}$, $\widehat{\boldsymbol{m}}_{N_{2,r}}$, $\boldsymbol{m}_{N_{2,r}}$ respectively, and since $p_r = p_{N_{2,r}}$ by proposition 6.1.3, we shall adopt the simpler notation p_r.

Let $S_E(\mathscr{P})$ denote the E-valued, measurable simple processes. That is,

$$S_E(\mathscr{P}) \;=\; \{K = \sum_{i=1}^{n} e_i 1_{A_i} : n \in \mathbb{N}, e_i \in E, \text{ and } A_i \in \mathscr{P}\}\ .$$

A process $H : \mathbb{R}^+ \times \Omega \to E$ is *measurable* if it is the pointwise limit of a sequence $\{K_n\} \subset S_E(\mathscr{P})$. Given any $r \in \mathscr{G}$, we define the function $\mathfrak{N}_r$ on the measurable processes H by

$$\mathfrak{N}_r(H) \;=\; \sup_{z \in U^\circ_{N_{2,r}}} \int p_r(H(t,\omega))\, \mathrm{d}\,|\,\boldsymbol{m}_z\,|_r\ .$$

We have that, for each $r \in \mathscr{G}$, $\mathfrak{N}_r$ is a seminorm on

$$\mathfrak{F} \;=\; \mathfrak{F}(\boldsymbol{m}_{E,L^2_G}) \;=\; \{\text{measurable } H : \mathfrak{N}_r(H) < \infty \;\text{ for each }\; r \in \mathscr{G}\}\ ,$$

and $(\,\mathfrak{F}(\boldsymbol{m}_{E,L^2_G})\,,\,\{\mathfrak{N}_r\}\,)$ forms a locally convex space. The space $\mathfrak{L} = \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$ is the closure of the space of measurable simple functions $S_E(\mathscr{P})$ in $\mathfrak{F}$.

We have the following definitions:

Definition 6.2.1. For $H, H_1, H_2, \ldots$ measurable, we say:

(1) $\{H_n\}$ is *mean Cauchy* if $\mathfrak{N}_r(H_n - H_m) \overset{m,n}{\to} 0$ for each $r \in \mathscr{G}$.

(2) $H_n \to H$ *in mean* if $\mathfrak{N}_r(H_n - H) \overset{n}{\to} 0$ for each $r \in \mathscr{G}$.

(3) $\{H_n\}$ is *Cauchy in $\boldsymbol{m}_{E,L^2_G}$-measure* if, for each $r \in \mathscr{G}$ and $\epsilon > 0$, we have

$\widehat{\boldsymbol{m}}_r \{ r(H_n - H_m) > \epsilon\} \overset{m,n}{\to} 0.$

(4) $H_n \to H$ *in $\boldsymbol{m}_{E,L^2_G}$-measure* if, for each $r \in \mathscr{G}$ and $\epsilon > 0$, we have

$\widehat{\boldsymbol{m}}_r\{ r(H_n - H) > \epsilon\} \overset{n}{\to} 0.$

Theorem 6.2.2. *Let $H : \mathbb{R}^+ \times \Omega \to E$ be a measurable process. Then $H \in \mathfrak{L}$ if and only if H has a determining sequence $\{H_n\} \subset S_E(\mathscr{P})$, defined by*

(1) *$\{H_n\}$ is mean Cauchy.*

(2) *$H_n \to H$ in $\boldsymbol{m}_{E,L^2_G}$-measure.*

Moreover, given any sequence $\{H_n\} \subset \mathfrak{L}$, the following are equivalent:

(1) *$\{H_n\}$ is mean Cauchy and $H_n \to H$ in $\boldsymbol{m}_{E,L^2_G}$-measure.*

(2) *$H \in \mathfrak{L}$ and $H_n \to H$ in mean.*

For $H = \sum_{i=1}^n e_i 1_{A_i} \in S_E(\mathscr{P})$ and any $A \in \mathscr{P}$, we have:

$$\int_A H \,\mathrm{d}\boldsymbol{m} \quad = \quad \sum_{i=1}^{n} \overline{\phi}(e_i)\boldsymbol{m}(A \cap A_i) \quad = \quad \int H 1_A \,\mathrm{d}\boldsymbol{m} \;.$$

More generally, for any $H \in \mathfrak{L}$, we have that

$$\int_A H \,\mathrm{d}\boldsymbol{m} \quad = \quad \lim_{n\to\infty} \int_A H_n \,\mathrm{d}\boldsymbol{m}$$

for any determining sequence $\{H_n\}$, and where the limit is taken in L^2_G. We adopt the notation

$$\int_A H \,\mathrm{d}X \quad := \quad \int_A H \,\mathrm{d}\boldsymbol{m} \;,$$

which emphasizes that $\boldsymbol{m} = I_X$ is produced by the square-integrable martingale X. The set function $\int_{(\cdot)} H \,\mathrm{d}X$ is well-defined, and is σ-additive on $\mathscr{P}$ since $\boldsymbol{m}$ is σ-additive.

Furthermore, we have the following generalization, in accordance with theorem 4.2.11.

Theorem 6.2.3. *Let $\{H_n\} \subset \mathfrak{L}$ and $H \in \mathfrak{L}$. We have*

$$N_r\Big(\int_A H \,\mathrm{d}\boldsymbol{m}\Big) \quad \le \quad \mathfrak{N}_r(H1_A) \;,$$

for each $A \in \mathscr{S}$ and $r \in \mathscr{G}$. If $H_n \overset{n}{\to} H$ in $\mathfrak{L}$, then for each $A \in \mathscr{S}$,

$$\int_A H_n \,\mathrm{d}\boldsymbol{m} \quad \overset{n}{\to} \quad \int_A H \,\mathrm{d}\boldsymbol{m} \quad \textit{in } L^2_G \;.$$

Finally, we have the following convergence theorems.

Theorem 6.2.4. *Let* $\{H_n\} \subset \mathfrak{L}$, *and let* $H : \mathbb{R}^+ \times \Omega \to E$ *be a measurable process. Suppose*

(1) $H_n \to H$ *in* $\boldsymbol{m}_{E,L^2_G}$*-measure.*

(2) $\lim_{\widehat{\boldsymbol{m}}_r(\cdot) \to 0} \mathfrak{N}_r(H_n 1_{(\cdot)}) = 0$, *uniformly in* n, *for each* $r \in \mathscr{G}$.

Then $H \in \mathfrak{L}$ *and* $H_n \to H$ *in mean. Conversely, if* $H_n \to H$ *in* $\mathfrak{L}$, *then properties* (1) *and* (2) *hold.*

Theorem 6.2.5. *Let* $\{H_n\} \subset \mathfrak{L}$ *and* $K \in \mathfrak{L}$. *Let* H *be a measurable process. Suppose*

(1) $H_n \to H$ *in* $\boldsymbol{m}_{E,L^2_G}$*-measure.*

(2) *For each* $r \in \mathscr{G}$, *we have* $p_r(H_n) \leq p_r(K)$ *for all* $n \in \mathbb{N}$.

Then $H \in \mathfrak{L}$ *and* $H_n \to H$ *in mean.*

Additionally, recall that E, G are nuclear locally convex spaces with bases of seminorms $\mathscr{E}, \mathscr{G}$ satisfying the conditions of Ustunel, and that $\mathscr{G}$ is countable, so that L^2_G inherits the countable basis of seminorms $\{N_{2,r}\}$ where the r range over $\mathscr{G}$. For each $r \in \mathscr{G}$, the space $L^2_G(U_{N_{2,r}})$ is Hilbert by theorem 5.4.2, since $G(U_r)$ is Hilbert, and therefore contains no copy of c_o. Hence, by theorem 4.4.4, we have the following result.

Theorem 6.2.6. *Suppose* $H : \mathbb{R}^+ \times \Omega \to E$ *is a bounded measurable process. Then* $H \in \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$.

We now turn our attention once more to the alternate method of defining the integral of a bounded measurable process.

The current assumptions on E, F, G satisfy the requirements of remark 4.4.2. In particular, given a bounded measurable process $H : \mathbb{R}^+ \times \Omega \to E$, we may obtain closed convex balanced bounded sets B, B_o from G, E respectively, such that the set B_o contains the range of H, the set B contains $\Phi(B_o, F_1)$, and each of $E[B_o], G[B]$ is a separable Hilbert space. This time, however, we form the map $\overline{\phi}_o : E[B_o] \to L(L^2_F, L^2_{G[B]})$ in the usual way, and note that the $\overline{\phi}_o$-semivariation $\widehat{\boldsymbol{m}}_{E[B_o],L^2_{G[B]}}$ is finite by proposition 3.2.4.

Now, algebraically, we have that H takes its values in $E[B_o]$. Since $H : T \to E$ is assumed bounded and measurable with respect to the topology of E, then lemma 4.4.1 shows that $H : T \to E[B_o]$ is still bounded and measurable. Thus, it is integrable relative to

$\overline{\phi}_o : E[\,B_o\,] \to L(\,L^2_F, L^2_{G[\,B\,]}\,)$. That is, we could define

$$\int H \,\mathrm{d}X \;:=\; \int H \,\mathrm{d}\boldsymbol{m} \;\in\; L^2_{G[B]}\,,$$

and the process $\int H \,\mathrm{d}X$ defined by

$$\Big(\int H \,\mathrm{d}X\Big)_t \;=\; \int H 1_{[0,t]} \,\mathrm{d}X$$

is then a square-integrable martingale by theorem 3.2.7.

Once again, it is important to remember that the choices of B and B_o were dependent upon the H under consideration. However, in parallel fashion to remark 4.4.5, we make an observation concerning the proof of theorem 6.2.6 (which is the proof of theorem 4.4.4), as it pertains to the current context. Regardless of the choice of B_o containing the range of H, and B for which $\Phi(B_o, F_1) \subset B$, we could have taken λ to be $\lambda = \mu + \mu'$ where $M_{E,L^2_G} \ll \mu$ and $\widehat{\boldsymbol{m}}_{E[B_o],L^2_{G[B]}} \ll \mu'$. Then, the sequence $\{H_n\}$ obtained in the proof is simultaneously shown to be a determining sequence for H in $\mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$ and a determining sequence for H in $\mathfrak{L}(\boldsymbol{m}_{E[B_o],L^2_{G[B]}})$, that is, relative to the embedding $\overline{\phi}_o : E[B_o] \to L(\,L^2_F, L^2_{G[B]}\,)$ of Hilbert spaces. We obtain integrals $\int H \, dX$ in each case. The latter we denote by K_1 and the former by K_2. By theorem 5.4.2, $L^2_{G[B]} \subset L^2_G$, where $L^2_{G[B]}$ consists of measurable functions, and the embedding is continuous with respect to the topologies involved. Thus,

$$\int H_n \, dX \to K_1 \in L^2_{G[B]} \subset L^2_G,$$

and

$$\int H_n \, dX \to K_2 \in L^2_G,$$

so that the integrals $K_1 = K_2$, which happen to take values in the span of B, are equal in L^2_G. Since this is true regardless of the choice of B_o, B, we see that the integrals for H obtained relative to the various embeddings $\overline{\phi}_o : E[B_o] \to L(\,L^2_F, L^2_{G[B]}\,)$ by this alternate approach for bounded H are all identical, and in fact, identical to the integral obtained by regarding $H \in \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$ in the original, more general approach for all functions in $\mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$.

6.3 Martingales

Recall the definition 3.2.2 of a square-integrable martingale $M : \mathbb{R}^+ \times \Omega \to F$, where F is a Hilbert space. In that section, theorem 3.2.7 demonstrates that when X is a square-integrable martingale, $H : \mathbb{R}^+ \times \Omega \to E$ is an integrable process, E is Banach space, and G is a Hilbert space, we have that $\int H \, \mathrm{d}X : \mathbb{R}^+ \times \Omega \to G$ is a square-integrable martingale when we define

$$\Big[\int H \, \mathrm{d}X \Big] (t, \omega) \;=\; \Big[\int H 1_{[0,t]} \, \mathrm{d}X \Big] (\omega) \;=\; \Big[\int H \, \mathrm{d}X \Big]_t (\omega) \,,$$

for every $t \geq 0$ and $\omega \in \Omega$.

We wish to extend this notion to the context of section 6.2 in a natural way. In this setting, we have a square-integrable martingale $X : \mathbb{R}^+ \times \Omega \to F$, where F is a Hilbert space. Given an integrable process $H : \mathbb{R}^+ \times \Omega \to E$, we have that, for each $A \in \mathscr{S}$,

$$\int_A H \, \mathrm{d}X \;=\; \int H 1_A \, \mathrm{d}X \;\in\; L^2_G(\mathcal{P}) \,. \tag{6.5}$$

Therefore we make the definition:

Definition 6.3.1. We say a process $Y : \mathbb{R}^+ \times \Omega \to G$, where G is a locally convex space, is a *square-integrable martingale* if there is a random variable $Y_\infty \in L^2_G = L^2_G(\mathcal{P})$ such that $Y_t = E(Y_\infty | \mathscr{F}_t)$ for all $t \geq 0$. That is, for each $t \geq 0$ and every $A \in \mathscr{F}_t$,

$$\int_A Y_t \, \mathrm{d}\mathcal{P} \;=\; \int_A Y_\infty \, \mathrm{d}\mathcal{P} \,,$$

where $Y_t(\omega) = Y(t, \omega)$ for all $t \geq 0$ and all $\omega \in \Omega$.

Note that above definition is analogous to the second, equivalent characterization of a square-integrable martingale given in definition 3.2.2. As above, we define the process $H \cdot X = \int H \, \mathrm{d}X : \mathbb{R}^+ \times \Omega \to G$ for an integrable process H to be

$$\Big[\int H \, \mathrm{d}X \Big] (t, \omega) \;=\; \Big[\int H 1_{[0,t]} \, \mathrm{d}X \Big] (\omega) \;=\; \Big[\int H \, \mathrm{d}X \Big]_t (\omega) \,, \tag{6.6}$$

for every $t \geq 0$ and $\omega \in \Omega$. By (6.5), we have $(H \cdot X)_t \in L^2_G$ for every $t \geq 0$, and additionally, we see that the natural candidate for $(H \cdot X)_\infty$ would be the process

$$(H \cdot X)_\infty \;=\; \int H 1_{[0,\infty)} \, \mathrm{d}X \; .$$

In order to achieve this result, we first prove two lemmas. The first will allow us to obtain results by examining simple functions that involve characteristic functions of the form $1_{(s,t] \times A}$, where $0 < s \leq t < \infty$ and $A \in \mathscr{F}$, instead of those of the form 1_B where B lies in the more complicated family $\mathscr{P} = \sigma(\mathscr{R})$. The second lemma is a technical result that will allow us to pass the linear map $\phi(e) \in L(F, G)$ through the F-valued stochastic integral of an element of L^2_F; this is nontrivial, since G is a locally convex space.

Lemma 6.3.2. *Given any* $e1_A \in S_E(\mathscr{P})$, *where* $e \in E$ *and* $A \in \mathscr{P}$, *there exists a sequence of sets* $\{A_n\} \subset \mathscr{R}$ *for which the sequence of functions* $\{e1_{A_n}\} \subset S_E(\mathscr{R})$ *satisfies* $e1_{A_n} \xrightarrow{n} e1_A$ *in* $\mathfrak{L}(m_{E,L^2_G})$.

Proof. The process $e1_A$ is bounded and measurable. We utilize the control measure λ, the closed convex balanced bounded set B_o containing the range of $e1_A$, and the other general details of the proof of theorem 6.2.6 (which is the proof of theorem 4.4.4), as it pertains to the current setting. The norm of the space $E[B_o]$ will be simply denoted by $\|\cdot\|$. Since λ is positive and σ-additive on $\mathscr{P} = \sigma(\mathscr{R})$, there is a sequence of $A_n \in \mathscr{R}$ for which $\lambda(A_n \Delta A) \xrightarrow{n} 0$. But this means first of all that

$$\begin{aligned} \int \| e1_{A_n} - e1_A \| \, \mathrm{d}\lambda \;&=\; \int \| e1_{A_n \Delta A} \| \, \mathrm{d}\lambda \\ &=\; \| e \| \, \lambda(A_n \Delta A) \;\xrightarrow{n}\; 0 \, , \end{aligned}$$

and secondly that, for any $\epsilon > 0$,

$$\lambda \{ \| e1_{A_n} - e1_A \| > \epsilon \} \;=\; \lambda \{ \| e1_{A_n \Delta A} \| > \epsilon \}$$

converges to 0 in n. Finally, putting $A = A_0$, note that the values $\| e1_{A_n}(t, \omega) \|$ are uniformly bounded by $\| e \|$ on $\mathbb{R}^+ \times \Omega$ for $n = 0, 1, 2, \ldots$ These three properties suffice in order to follow the remainder of the proof to the conclusion that $\{e1_{A_n}\}$ is a determining sequence for $e1_A$, and hence that $e1_{A_n} \xrightarrow{n} e1_A$ in $\mathfrak{L}(m_{E,L^2_G})$. ☐

Lemma 6.3.3. *Let* $e \in E$ *and* $K \in L^2_F$. *Then*

$$\int \overline{\phi}(e)K \, \mathrm{d}\mathcal{P} \quad = \quad \phi(e) \int K \, \mathrm{d}\mathcal{P}.$$

Proof. First suppose that $K = f1_A$, where $f \in F$ and $A \in \mathscr{F}$. We have

$$\begin{aligned} \phi(e) \int f1_A \, \mathrm{d}\mathcal{P} &= \phi(e)(\mathcal{P}(A)f) \\ &= \mathcal{P}(A)\phi(e)f \\ &= \int [\phi(e)f]1_A \, \mathrm{d}\mathcal{P} \quad = \quad \int \overline{\phi}(e)(f1_A) \, \mathrm{d}\mathcal{P} \,. \end{aligned}$$

Since $\phi(e)$ and $\overline{\phi}(e)$ are linear maps, we may extend the result to all simple functions $K \in S_F(\mathscr{F})$ by the additivity of the integral.

For a general $K \in L^2_F$, we obtain a sequence of simple functions $\{K_n\}$ for which $K_n \xrightarrow{n} K$ in L^2_F. By the continuity of $\overline{\phi}(e) \in L(L^2_F, L^2_G)$, we have that $\overline{\phi}(e)K_n \xrightarrow{n} \overline{\phi}(e)K$ in L^2_G. Now regarding $\{K_n\}$ and $\{\overline{\phi}(e)K_n\}$ as determining sequences, we have

$$\int K \, \mathrm{d}\mathcal{P} \quad = \quad \lim_{n\to\infty} \int K_n \, \mathrm{d}\mathcal{P} \ \text{ in } F \,.$$

and

$$\int \overline{\phi}(e)K \, \mathrm{d}\mathcal{P} \quad = \quad \lim_{n\to\infty} \int \overline{\phi}(e)K_n \, \mathrm{d}\mathcal{P} \ \text{ in } G \,.$$

Then, by the continuity of $\phi(e) \in L(F, G)$,

$$\begin{aligned} \phi(e) \int K \, \mathrm{d}\mathcal{P} &= \phi(e) \, (\lim_{n\to\infty} \int K_n \, \mathrm{d}\mathcal{P}) \\ &= \lim_{n\to\infty} \int \overline{\phi}(e)K_n \, \mathrm{d}\mathcal{P} \\ &= \int \overline{\phi}(e)K \, \mathrm{d}\mathcal{P} \,. \end{aligned}$$

Thus, the result has been demonstrated. □

We now prove that the process $H \cdot X$ given in (6.6), which is produced from the integral of an integrable process H, is a square-integrable martingale as characterized by definition 6.3.1.

Theorem 6.3.4. *Let* $H \in \mathfrak{L}(\boldsymbol{m}_{E,L^2_G})$. *Then the process* $H \cdot X$ *is a square-integrable martingale.*

Proof. As noted earlier in (6.5), we have that $\int H \,\mathrm{d}X \in L^2_G$ for all $A \in \mathscr{P}$. In particular, choose $A = [\,0,\infty\,) \times \Omega$. Put

$$(H \cdot X)_\infty \;:=\; \int H 1_{[\,0,\infty\,)} \,\mathrm{d}X \;\in\; L^2_G \,.$$

For brevity, we denote $Y = H \cdot X$, so that for each $t \geq 0$,

$$Y_t \;:=\; \int H 1_{[\,0,t\,]} \,\mathrm{d}X \;\in\; L^2_G \,.$$

We claim that $Y_t = E(Y_\infty | \mathscr{F}_t)$. That is, for each $t \geq 0$ and every $A \in \mathscr{F}_t$, we must show that

$$\int_A Y_t \,\mathrm{d}\mathcal{P} \;=\; \int_A Y_\infty \,\mathrm{d}\mathcal{P} \,.$$

That is, using the additivity of the integral, we must show that, for each $t \geq 0$ and every $A \in \mathscr{F}_t$,

$$\int 1_A \int H 1_{(\,t,\infty\,)} \,\mathrm{d}X \,\mathrm{d}\mathcal{P} \;=\; 0 \,. \tag{6.7}$$

At first, consider $H = e1_{C\times B}$ where $e \in E$ and either of the following hold:

(1) $C = \{0\}$ and $B \in \mathscr{F}_0$.

(2) $C = (\,v, w\,]$ where $0 \leq v < w \leq t$ and $B \in \mathscr{F}_v$.

Then the integrand in (6.7) is identically zero, and thus the equation is trivial.

Next, consider $H = e1_{(\,v,w\,]\times B}$, where $t < w$ and $B \in \mathscr{F}_v$. Then,

$$\begin{aligned}\int H 1_{(\,t,\infty\,)} \,\mathrm{d}X &= \overline{\phi}(e)\, I_X(\,(\,\max\{t,v\}, w\,] \times B\,) \\ &= 1_B \overline{\phi}(e)(X_w - X_{\max\{t,v\}}) \,,\end{aligned}$$

and thus, due to the fact that $A \cap B \in \mathscr{F}_{\max\{t,v\}}$ and X is a martingale,

$$\begin{aligned} 0 &= \phi(e) \int_{A\cap B} (X_w - X_{\max\{t,v\}}) \,\mathrm{d}\mathcal{P} \\ &= \int_{A\cap B} \overline{\phi}(e)(X_w - X_{\max\{t,v\}}) \,\mathrm{d}\mathcal{P} \quad \text{by lemma 6.3.3,} \\ &= \int 1_A \int H 1_{(\,t,\infty\,)} \,\mathrm{d}X \,\mathrm{d}\mathcal{P} \,. \end{aligned}$$

Thus, (6.7) holds for every $t \geq 0$ when H is of the form $e1_B$, where $e \in E$ and B is a predictable rectangle. By the additivity of the integral, and by writing any element of $\mathscr{R}$ as a finite union of disjoint rectangles, we see that (6.7) holds for every $t \geq 0$ when $H \in S_E(\mathscr{R})$.

We now apply lemma 6.3.2. Given a process $H = e1_B$, where $e \in E$ and $B \in \mathscr{P}$, there exists a sequence of $B_n \in \mathscr{R}$ for which the simple functions $H_n := e1_{B_n} \overset{n}{\to} H$ in $\mathfrak{L}(m_{E,L^2_G})$. Thus, $\{H_n\}$ is a determining sequence for H, so that by definition of the indefinite integral,

$$\int H_n 1_{(t,\infty)} \, \mathrm{d}X \overset{n}{\to} \int H 1_{(t,\infty)} \, \mathrm{d}X \quad \text{in } L^2_G \, .$$

As shown in section 5.3, the above convergence occurs in L^1_G as well. Then, by theorem 5.1.3,

$$\int 1_A \int H_n 1_{(t,\infty)} \, \mathrm{d}X \, \mathrm{d}\mathcal{P} \overset{n}{\to} \int 1_A \int H 1_{(t,\infty)} \, \mathrm{d}X \, \mathrm{d}\mathcal{P} \quad \text{in } G \, .$$

But, the left side is 0 for every $n \in \mathbb{N}$, and thus, (6.7) holds for processes of the form $H = e1_B$, where $e \in E$ and $B \in \mathscr{P}$. Once again, by the additivity of the integrals, we see that (6.7) holds for $H \in S_E(\mathscr{P})$.

Finally, for a general $H \in \mathfrak{L}(m_{E,L^2_G})$, we may obtain a sequence $\{H_n\} \subset S_E(\mathscr{P})$ that converges to H in $\mathfrak{L}(m_{E,L^2_G})$. Applying the same reasoning as in the preceding paragraph, we have that (6.7) is satisfied by $H \in \mathfrak{L}(m_{E,L^2_G})$. $\square$

REFERENCES

[B–P] Bessaga C., Pelczynski A. (1974), *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **5**, 151–164.

[B] Bochner S. (1933), *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fund. Math. **20**, 262–276.

[Br1] Brooks J. K. (1995), *Stochastic processes in Banach spaces, quasimartingales and stochastic integrals*, Rend. Mat. Trieste, XXVI, 7–33.

[Br2] Brooks J. K. (1999), *Stochastic integration in nuclear spaces*, University of Florida, Preprint.

[Br3] Brooks J. K. (2002), *Stochastic integration theory in infinite-dimensional spaces*, Func. Anal. VII, Aarhus **46**, 83–95.

[Br4] Brooks J. K. (2002), *Stochastic processes and stochastic integration in Banach spaces*, Handbook of Measure Theory, Chapter 10 **1**, 449–503.

[B–D1] Brooks J. K., Dinculeanu N. (1974), *Strong additivity, absolute continuity and compactness in spaces of measures*, J. Math. Anal. Appl. **45**, 156–175.

[B–D2] Brooks J. K., Dinculeanu N. (1976), *Lebesgue-type spaces for vector integration, linear operations, weak completeness and weak compactness*, J. Math. Anal. Appl. **54**, 348–389.

[B–D3] Brooks J. K., Dinculeanu N. (1991), *Stochastic integration in Banach spaces*, Seminar on Stochastic Processes, Birkhäuser, Basel, 27–115.

[B–N] Brooks J. K., Neal D. (1998), *Generalized stochastic integral on nuclear spaces*, Atti. Sem. Mat. Fis. Modena **46**, 83–98.

[D–M] Déllacherie C., Meyer P. A. (1978), *Probabilities and Potential*, North-Holland, Amsterdam.

[D1] Dinculeanu N. (1967), *Vector Measures*, Pergamon Press, New York.

[D2] Dinculeanu N. (2000), *Vector Integration and Stochastic Integration in Banach Spaces*, Wiley-Interscience, New York.

[D–S] Dunford N., Schwartz J. T. (1957), *Linear Operators Part I*, Interscience, New York.

[H] Halmos P. R. (1974), *Measure Theory, Graduate Texts in Mathematics*, Springer–Verlag, New York.

[P] Pettis B. J. (1938), *On integration in vector spaces*, Trans. Am. Math. Soc. **44**(2), 277–304.

[Pi] Pietsch A. (1972), *Nuclear Locally Convex Spaces*, Springer–Verlag, Berlin–Heidelberg–New York.

[R–R] Robertson A. P., Robertson W. J. (1964), *Topological Vector Spaces*, Cambridge University Press, London.

[S] Schaefer H. H. (1970), *Topological Vector Spaces, Graduate Texts in Mathematics*, Springer–Verlag, Berlin–Heidelberg–New York.

[T] Treves F. (1967), *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York–London.

[U] Ustunel S. (1982), *Stochastic integration on nuclear spaces and its applications*, Ann. Inst. H. Poincaré **18**(2), 165–200.

BIOGRAPHICAL SKETCH

Jason T. Kozinski was born in Columbia, Missouri, and spent the majority of his childhood in various university towns in Missouri and Oklahoma. He moved to Florida in 1993 and was admitted to the University of Florida in 1995 as a National Merit Scholar, earning his B.S. and M.S. degrees in pure mathematics in 1998 and 2001, respectively. Jason is the youngest of four brothers, and his hobbies include music, art, poetry, sports, and home improvement.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

J K Brooks

James K. Brooks, Chair
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Scott McCullough
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Louis Block

Louis Block
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Li-Chien Shen
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Jeri Benson

Jeri Benson
Associate Professor of Educational Psychology

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Dean, Graduate School